Transforming Luxuries into Necessities: How Inequality Affects Growth?[*]

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Abstract

Using barcode-level data linked to firm accounts, we document that: firms selling more to richer households innovate more; these “luxury” goods later spread to lower-income consumers as prices fall and entry expands, becoming “necessities”; and greater inequality depresses innovation among firms exposed to low-income demand. To rationalize these patterns, we develop an endogenous growth model with hierarchical demand, incumbent R&D, and endogenous entry. Rising inequality reduces the purchasing power of the poor, which can distort innovation away from necessities and slow growth. When calibrated to U.S. data, the model implies sizable growth and welfare gains from redistribution.

Keywords: Inequality, Innovation, Non-homothetic preference, Growth.
JEL codes: H54, N45,O11, O16, O38, O53, O57.

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[*] Researchers own analyses calculated (or derived) based in part on data from Nielsen Consumer LLC and marketing databases provided through the NielsenIQ Datasets at the Kilts Center for Marketing Data Center at The University of Chicago Booth School of Business. The conclusions drawn from the NielsenIQ data are those of the researchers and do not reflect the views of NielsenIQ. NielsenIQ is not responsible for, had no role in, and was not involved in analyzing and preparing the results reported herein. We thank Yucheng Yang for sharing the GS1 Global Data. We thank Ziyue Wang for her excellent research assistance. We thank seminar participants at PKU and BUAA. Financial support from the Natural Science Foundation of China (Grant No. 72322007;72503230) is gratefully acknowledged by Mingzhi Xu.

"The capitalist achievement does not typically consist in providing more silk stockings for queens but in bringing them within the reach of factory girls in return for steadily decreasing amounts of effort."

– Joseph A. Schumpeter, *Capitalism, Socialism & Democracy* (1942)

1 Introduction

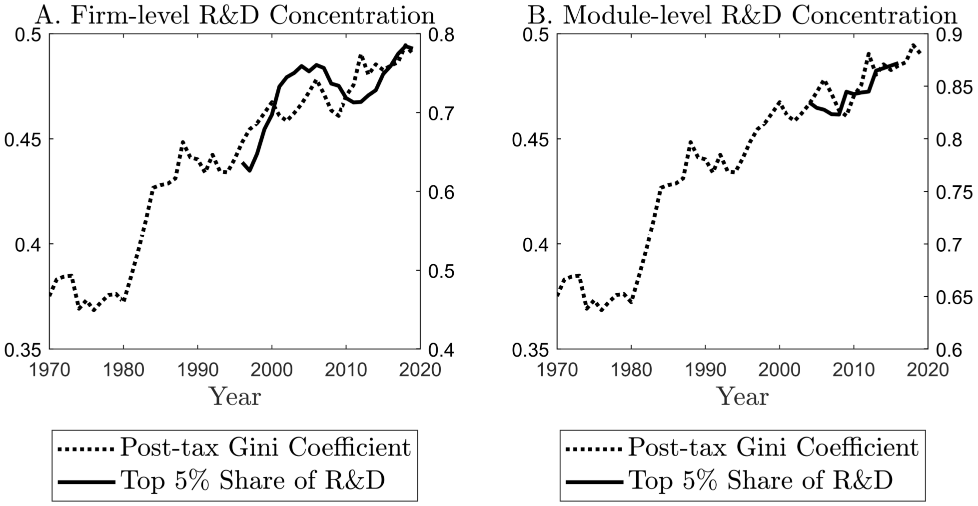
Rising income inequality is not only about who gets what, but also shapes what gets invented, who benefits from new technologies, and how much growth innovation delivers (Jones and Kim, 2018; Aghion et al., 2019; De Loecker et al., 2020). New products often begin as high-end goods targeted at affluent consumers and only gradually become widely used necessities as costs fall and markets expand (Matsuyama, 2002; Foellmi and Zweimüller, 2006). This transformation suggests that firms serving different segments of the market may face fundamentally different innovation incentives. A firm developing high-end robotic technologies for affluent consumers and a firm producing basic consumer goods such as shampoo are likely to respond very differently to rising income inequality in their innovation decisions. These observations raises several questions: how does rising inequality reshape the allocation of innovations across firms and sectors? What are the aggregate consequences for economic growth? What is the impact of redistribution policy through tax and subsidy on economic growth and welfare?

This paper attempts to answer these questions by combining empirical, theoretical, and quantitative analyses using an endogenous growth model. We show that inequality distorts innovation: it pulls R&D toward premium, rich-serving products and away from productivity gains in basic goods, weakening innovation-led growth. We call this the misallocation channel of inequality.

The misallocation channel emerges through the interaction of hierarchical demand and the product life cycle. New varieties are introduced at high prices and initially sold to affluent consumers. As innovation and entry reduce costs, prices fall, markets expand, and productivity gains diffuse. Higher inequality weakens this engine by compressing demand and profits in mass-market (necessity) segments, where broad-based cost reductions would otherwise generate the largest social returns. At the same time, stronger top-end demand raises the private return to innovation targeted at premium segments, encouraging a more concentrated allocation of R&D. The resulting tilt leaves too little innovative effort in basic goods and lowers the aggregate return to innovation.

This concentration is visible in U.S. aggregate time series. Figure 1 plots the U.S. post-tax Gini coefficient from the World Inequality Database together with two measures of where private R&D is carried out, constructed from BvD Orbis firm accounts (available annually from 1996 onward). Panel A reports the share of total U.S. R&D expenditure accounted for by the top 5% of firms ranked by R&D spending in each year. Panel B reports the analogous top-5% share across product modules, where a module is a fine-grained Nielsen classification that groups closely comparable items within a narrowly defined segment. As inequality rises, R&D becomes more concentrated in a small set of firms and product segments.

Figure 1: Income Inequality and the Concentration of R&D in the United States



Notes. The dotted line plots the U.S. post-tax Gini coefficient (left axis), from the World Inequality Database. Panel A overlays the share of aggregate R&D expenditure accounted for by the top 5% of R&D-spending firms (solid line, right axis), constructed from BvD Orbis (firm-level R&D available from 1996 onward). Panel B reports an analogous concentration measure computed across product modules: the share of total R&D attributable to the top 5% of modules. Higher values of the concentration series indicate that a larger fraction of economy-wide R&D is carried out by a small set of firms (Panel A) or a narrow set of product segments (Panel B).

To study this mechanism, we build a matched dataset linking household purchases and incomes to firm outcomes, product ownership, and patenting over the product life cycle. The dataset combines nationally representative household purchases with firm-level financial and patent data, allowing us to track how demand, innovation, and market structure evolve within narrowly defined product markets. Our empirical analysis delivers four stylized facts that characterize a common product life cycle and its link to innovation. First, products enter as high-priced varieties consumed primarily by high-income households and gradually diffuse to lower-income consumers as prices fall. Second, innovation is concentrated where early demand is strongest: firms serving higher-income consumers engage more intensively in patenting. Third, diffusion is accompanied by entry and declining market concentration, consistent with intensifying competition. Fourth, greater exposure to inequality is associated with weaker aggregate innovation, particularly in segments serving low-income consumers, and with a higher concentration of innovative activity.

Motivated by these facts, we build an endogenous growth model with heterogeneous households, hierarchical (non-homothetic) demand, and multiple margins of innovation. Households allocate spending across a continuum of goods ranked from necessities to higher-end products, as in Matsuyama (2002) and Foellmi and Zweimüller (2006). On the supply side, potential entrants choose whether to enter and how much to invest to raise their productivities (effectively reduce costs) when they become incumbents, while new varieties expand the frontier. Entry and cost-reducing innovation generate a natural product life cycle: goods arrive as high-priced luxuries, prices fall as productivity rises and competition intensifies, and adoption spreads down the income distribution. Entry increases and concentration declines along the same path, matching the diffusion and market-structure patterns in the data. The model delivers a tight link from inequality to the direction of innovation and long-run growth. Along the balanced growth path (BGP), luxury sectors sustain higher prices and higher innovation intensity, consistent with the cross-sector evidence. When income concentrates at the top, the mass market for necessities shrinks relative to premium demand, weakening incentives to innovate in broadly consumed goods. Innovative effort tilts toward rich-serving products, R&D becomes more concentrated, and the aggregate return to innovation falls, slowing growth.

Quantitatively, we calibrate the model to match the core module-level moments that define the luxury-to-necessity transition: the post-entry decline in prices, the gradual shift of adoption toward lower-income households, and the accompanying evolution of entry and concentration within a market. The calibrated economy also matches standard aggregates such as R&D intensity, firm-size concentration, and long-run growth. With the model disciplined by these facts, we use it to evaluate mechanisms and policy-relevant counterfactuals. A first decomposition shows that price declines and diffusion down the income distribution are driven in roughly equal parts by incumbents' cost-reducing R&D and by entry. Comparative statics that vary income inequality imply quantitatively important effects on growth, with magnitudes comparable to those observed in the historical U.S. experience. Finally, a progressive-tax reform that restores post-tax inequality to its 1970s level delivers large welfare gains: more than 99.5% of households benefit, with average welfare rising by about 70%. Therefore, rising inequality does more than shift demand: it misallocates R&D toward premium goods and away from cost reductions in necessities, where diffusion delivers the biggest gains. Policies that strengthen the mass market for basics can therefore redirect innovation toward broad growth and welfare.

Our paper makes three main contributions to the literature. First, we provide a comprehensive empirical characterization of the "from luxury to necessity" flying-geese pattern. While this pattern has been noted in the literature, existing studies typically rely on aggregate evidence or focus on specific margins. By contrast, we use comprehensive micro-level data to trace how consumption goods diffuse across consumers over the product life cycle, and to jointly document the evolution of innovation activity and firm dynamics along this process.

Our second contribution is to develop a unified theoretical framework that integrates multiple margins of innovation—process innovation by incumbents, innovation-driven entry that intensifies market competition, and the creation of new product varieties—and embeds them in a setting with non-homothetic demand. While each of these dimensions has been studied in isolation, relatively few papers incorporate all three within a unified framework. More importantly, linking these innovation margins to non-homothetic demand allows us to analyze how the distribution of income shapes firms' innovation incentives and the allocation of innovation across sectors.

Our third contribution is to uncover a novel misallocation channel through which inequality affects innovation and growth. While existing studies primarily emphasize price effects or market-size effects (Foellmi et al., 2014), we show that inequality shifts innovation resources toward luxury-oriented sectors and away from necessity-oriented sectors. This distortion has important aggregate consequences: as we demonstrate in the model and in the calibrated economy, slower technological progress in necessity sectors delays product diffusion to the mass market and slows down labor reallocation toward emerging sectors, thereby reducing overall economic growth.

Related Literature. Our theoretical model is most closely related to the literature on innovation in the presence of heterogeneous households with non-homothetic preferences. In these models, innovation raises household income and transforms goods that were once luxuries into necessities, exhibiting the so-called "flying-geese" pattern (Matsuyama, 2002; Foellmi and Zweimüller, 2006; Foellmi et al., 2014; Oberfield, 2023). In Matsuyama (2002) and Oberfield (2023), technological progress results from learning-by-doing, and long-run aggregate technological growth is either zero or exogenously determined. Foellmi and Zweimüller (2006) and Foellmi et al. (2014) examine how income inequality influences firms' endogenous innovation behavior. Foellmi and Zweimüller (2006) focus exclusively on innovation through the introduction of new goods, whereas Foellmi et al. (2014) extend the framework by additionally incorporating a one-time process innovation in each sector. However, none of these studies examines the heterogeneity of firms' R&D behavior across sectors during this diffusion process.

Compared with the existing literature, our study incorporates firm dynamics and emphasizes the heterogeneity of firms' R&D behavior, particularly how income inequality reshapes the allocation of R&D across sectors. Unlike prior models in which each consumption tier is served by a single monopolist, our framework features endogenous entry and competition among multiple firms producing the same good. Innovation and entry jointly push prices downward, allowing goods to diffuse from luxury to necessity status. This structure allows us to examine more clearly how income inequality differentially affects firms' innovation incentives across consumption tiers, and how the resulting reallocation of R&D across sectors shapes long-run economic growth. The framework is also particularly tractable and well-suited for quantitative analysis.

This perspective is also related to the literature on non-homothetic demand and the direction of technological progress. Recent studies show that innovation shifts toward more income-elastic sectors as mean income rises (Bohr et al., 2023; Comin et al., 2025). Our framework highlights a complementary mechanism: holding average income fixed, changes in the income distribution reweight effective market size across consumption tiers, thereby influencing the sectoral allocation of R&D. This mechanism helps organize the empirical patterns we document, including where patenting activity concentrates and how rapidly products transition from luxury to necessity status.

Our paper also contributes to the broader literature on inequality and growth. While evidence on the reduced-form relationship remains mixed (Persson and Tabellini, 1994; Forbes, 2000; Barro, 2000; Banerjee and Duflo, 2003; Baselgia and Foellmi, 2023), our mechanism operates through the direction of innovation. Inequality changes the composition of demand and, in turn, the relative profitability of cost reductions across sectors. In our model, a rise in inequality shifts innovative effort toward premium, rich-serving goods and away from cost reductions in necessities, where diffusion and scale generate broad productivity gains, lowering the aggregate return to innovation and slowing growth. A related literature studies innovation as a driver of inequality (Hémous and Olsen, 2022; Aghion et al., 2019). That work emphasizes how technological change reshapes the income distribution. Our analysis highlights the reverse feedback: the income distribution shapes innovation incentives by determining which markets are large enough to reward cost-reducing R&D and broad diffusion.

Our model mechanism is closely related to the demand-driven innovation approach of Foellmi and Zweimüller (2006), Foellmi et al. (2014), and Foellmi and Zweimüller (2017). These studies emphasize the price and market-size effects of innovations. Foellmi and Zweimüller (2006) show that higher inequality is favorable for growth as the price effect dominates the market-size effect. Foellmi et al. (2014) show that whenever process innovations—for which the market-size effect is of greater importance—are relevant for growth, excessively high inequality could be detrimental to growth, while an intermediate level of inequality renders maximal growth rate. Foellmi and Zweimüller (2017) find a similar result. Our paper identifies a new misallocation effect of inequality, which is detrimental to growth.

Finally, our paper builds on the endogenous-growth literature with firm dynamics. Pioneering contributions model productivity growth through innovations (Klette and Kortum, 2004; Lentz and Mortensen, 2008; Acemoglu et al., 2018) in the tradition of Romer (1990), Aghion and Howitt (1992), and Grossman and Helpman (1991). Subsequent work allows incumbents to invest in internal R&D alongside external innovation (Akcigit and Kerr, 2018; Peters, 2020; Akcigit and Ates, 2023). We extend this line by adding hierarchical demand and household heterogeneity, which makes market size by income group a determinant of entry, innovation incentives, and the sectoral allocation of R&D.

2 Empirical Analysis

We start from what households buy and then trace each product back to its supplier and forward to the supplier's innovation and financial outcomes. This linkage lets us follow a product's life cycle as it moves from high-income to broader consumers, and relate that diffusion to firms' innovative activity. In this section we first describe the data, and then lay out the key diffusion facts. We relegate additional descriptions and robustness check to Appendix A.

2.1 Data Description

The analysis draws on four primary sources. First, we use the NielsenIQ Homescan Consumer Panel, collected by NielsenIQ (U.S.) and made available through the Kilts Center for Marketing at the University of Chicago Booth School of Business. Second, we use GS1 Company Database to link the sample products obtained through NielsenIQ Homescan Consumer Panel to firms. Third, we incorporate the PatentsView dataset, compiled by the U.S. Patent and Trademark Office (USPTO). Finally, to obtain comprehensive firm-level and financial information, we draw on the Bureau van Dijk (BvD) Orbis database.

NielsenIQ Homescan Consumer Panel. We use the NielsenIQ Homescan Consumer Panel (NHCP) to obtain detailed information on household demographics, geographic location, and purchasing behavior. The panel, maintained and updated annually by the Nielsen Company, covers a geographically and demographically representative sample of roughly 40,000 to 60,000 U.S. households. Participating households record purchases using in-home scanners, providing information on the date of purchase, Universal Product Code (UPC), number of units, and total expenditure. To address potential selection bias, Nielsen constructs projection factors—sampling weights that scale to the U.S. household population—so that the dataset is projectable to the national level. In what follows, we introduce several key terms used throughout the analysis, define their empirical counterparts in the data, and explain how they map into variables in the model.

(i) Product Modules. Unless otherwise noted, we define a market in this paper as the set of transactions within a given product module. Each product module encompasses a variety of products that are defined at the most detailed level within the Nielsen data, with each item being distinctly identified by a 12-digit UPC paired with version codes. For ease of exposition, we use the terms "product" and "UPC" interchangeably to refer to this identifier throughout the analysis. We exclude so-called "Magnet" products that lack standard UPCs—items such as in-store baked goods—because only a subset of households report these purchases, and coverage is unavailable prior to 2007.

(ii) Firms. We link each UPC to its owning company using the GS1 Company Database. GS1 is a global not-for-profit standards organization that assigns barcodes to manufacturers worldwide. The database provides company names and GS1 prefixes—unique numerical codes embedded in the initial digits of each barcode—that connect UPCs to their parent firms (Hottman et al., 2016).[1] For instance, *Procter & Gamble* appears in our data as a single firm encompassing multiple brands, including *Crest* in oral hygiene and *Pantene and Head & Shoulders* in hair care. Restricting attention to GS1 companies with sales recorded in the Nielsen data during our sample period (2004–2016) yields 44,030 distinct firms.

To broaden the description of these firms and facilitate the estimates of firm-level productivity and markups, we incorporate business information from the Bureau van Dijk (BvD) Orbis database. Orbis provides comprehensive financial data on both listed and unlisted firms worldwide. We use a string-matching algorithm, supplemented by manual validation, to match GS1 company names with Orbis entries, achieving an 86.38% success rate. Consumer-focused industries such as grocery wholesaling and food manufacturing dominate, reflecting the prominence of firms producing daily consumer goods.

(iii) Consumer Groups. We classify consumers into groups based on annual household income, as reported in the Nielsen panel. Each participating household is asked to record its combined

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[1] For 16.53% of UPCs, no firm code could be matched between the GS1 data and the UPC list. These UPCs are excluded from the analysis. With 2.36 million products spanning categories such as food, non-food groceries, health and beauty aids, and general merchandise, the NHCP database accounts for about 66% of total products from U.S. GS1 firms, thereby ensuring the representativeness of our dataset.

total income, which Nielsen assigns to a predefined income range. A practical challenge arises because the top income category in the demographic survey changes over time, whereas the lower income ranges remain stable. For instance, in 2004–2005 the highest category was defined as "$100,000 or more." Beginning in 2006, Nielsen subdivided this range into four brackets: $100,000–124,999, $125,000–149,999, $150,000–199,999, and $200,000 or more. Following Nielsen's convention from 2010 onward, we collapse these upper ranges into a single top-income group to maintain consistency over time. Unless otherwise noted, we define "rich" consumers as those in the highest income category—roughly the top decile of the distribution—and "poor" consumers as those with income below $20,000, approximately the bottom decile. This harmonized definition ensures comparability across survey years while capturing meaningful variation at both ends of the income distribution.

(iv) Revenues and Prices. Firm revenue, used as a proxy for firm size, is computed as the sum of projection factor–weighted annual household expenditures on a firm's products. Product revenue is defined analogously, aggregating expenditures at the product level. Product prices are then calculated as the ratio of annual product revenue to projection factor–weighted units sold. To account for inflation, we express all prices in real terms using the U.S. Consumer Price Index (CPI) from the World Bank.

PatentsView Database. We measure firm-level innovation using detailed patent data from the PatentsView database, an official platform maintained by the U.S. Patent and Trademark Office (USPTO). PatentsView provides disambiguated assignee identifiers, allowing consistent linkage of patents to firms. Following the standard approach in the innovation literature, our baseline measure of innovation output is the number of patent applications filed by a firm in a given year that are ultimately granted (He and Tian, 2013). This measure captures the timing of inventive activity at the application stage while restricting attention to patents that pass the threshold of being granted, thus reflecting substantive technological advancement. To complement this measure, we also construct claim-weighted patent counts as an indicator of patent quality, since patents with more claims confer broader and more expansive protection (Aghion et al., 2019).

To link patents to firms, we employ a multi-step string-matching procedure that combines edit-distance algorithms with complementary methods and manual verification. The resulting matches closely reproduce the distribution of patenting activity in the USPTO universe: the lower tail aligns almost exactly, while the slight upward shift in the upper quartiles reflects the consolidation of assignee name variants into single firms. Geographic and firm-level patterns further validate the match, with patents concentrated in well-known innovation hubs such as California and Massachusetts and global technology leaders—LG Electronics, Microsoft, General Electric, and Apple—dominating the top of the distribution.

2.2 Motivational Facts

We first document empirical facts on the diffusion of new products across consumers with different income levels. We begin by describing how newly introduced products spread within the market, and then investigate potential explanations for these diffusion patterns. Specifically, we consider how innovation by incumbent firms, the entry of new competitors, and changes in the income distribution shape the process through which products move from high- to low-income consumers. Finally, we shed light on how shifts in the income distribution influence innovation distribution.

Fact I: Product Diffusion Along the Income Distribution. We measure product diffusion at the *module* level rather than at the UPC level. A module groups closely related UPCs that share similar physical attributes and serve the same consumption purpose, such as sparkling water, yogurt, or shampoo. Each module contains multiple UPCs that vary by brand, package size, or other variants. This aggregation reduces mechanical UPC-level churn while preserving substantial within-category variation, making it well suited for tracing diffusion patterns along the income distribution.[2]

We trace products' spread across the distribution – whether adoption begins with high-income households and diffuses downward, or vice versa. We examine this pattern by estimating

$$D_{fmt} = \mu_y \times f_{my} + \eta_{fm} + \eta_{ft} + \eta_{mt} + \epsilon_{fmt}. \tag{1}$$

The dependent variable, D_{fmt}, captures the diffusion of module m sold by firm f in year t. We measure D_{fmt} in two complementary ways: (i) the income position of the module's buyers, computed as the projection-factor-weighted mean income quantile, and (ii) the module's expenditure share accounted for by lower-income households (income below \$20,000).[3] The key regressor, f_{my}, is event time relative to entry, defined as the number of years since firm f first sells any product in module m during the sample period. The specification includes a rich set of fixed effects. Firm–module fixed effects (η_{fm}) absorb persistent differences in product attributes and firm positioning within a module. Firm–year fixed effects (η_{ft}) absorb time-varying firm-wide shocks, such as changes in brand reputation or marketing intensity. Module–year fixed effects (η_{mt}) absorb common shocks

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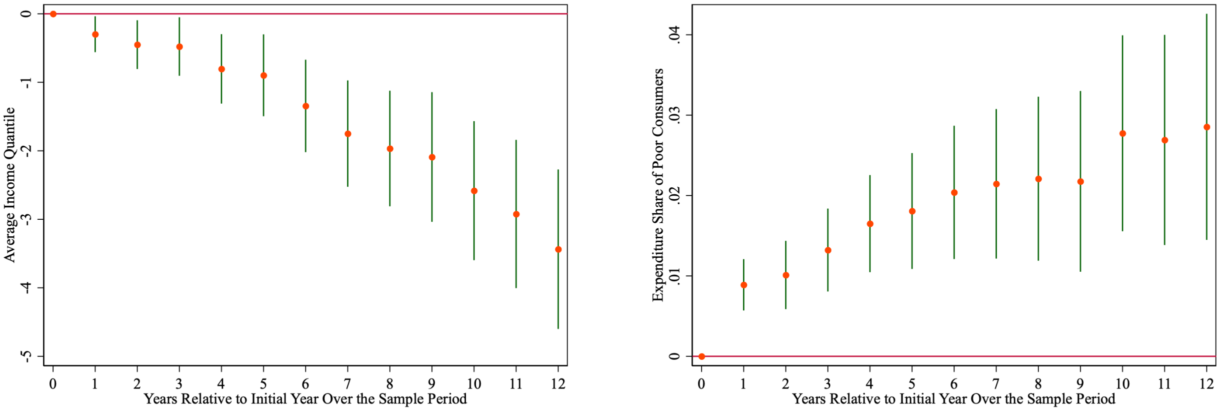
[2] This aggregation is motivated by three considerations. First, UPCs are a noisy proxy for innovation, since barcode changes often reflect packaging revisions, bundle adjustments, or temporary promotions rather than genuine product development, mechanically inflating product entry and exit. Aggregating to the module level filters out this noise and yields a more stable measure of product availability and adoption. Second, although prices and promotions are set at the UPC level, firms typically manage innovation and product upgrading at a broader level, so new lines or quality improvements usually affect clusters of related UPCs rather than a single barcode (Braguinsky et al., 2021). Third, our interest is in diffusion across income groups. The relevant margin is whether a product type becomes accessible to lower-income households, not which specific UPC they choose within that category. Module-level adoption therefore better captures the shift from non-consumption to consumption, and more broadly, the movement of a good from luxury to necessity. This view is consistent with Ma et al. (2025), who show that UPC-level purchases are highly volatile and often distorted by minor stock-keeping changes.

[3] We weight households using Nielsen's *projection factor* and compute income quantiles by year. This reweighting aligns the income distribution in the Nielsen sample with that of the U.S. population.

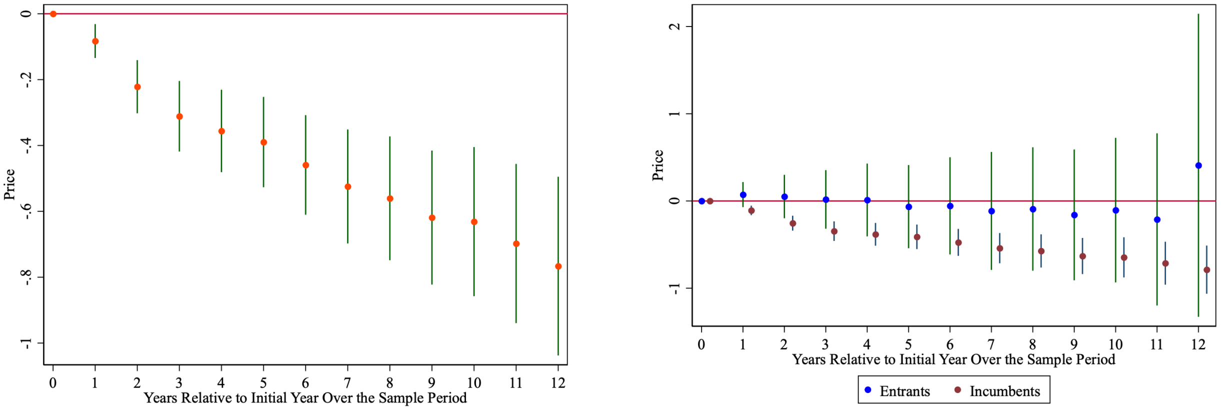
within a module-year, including shifts in tastes or technology. Standard errors are clustered at the firm level to allow arbitrary correlation within firms over time.

Figure 2: Product Diffusion Across the Income Distribution



(a) Decline in Buyers' Income

(b) Growing Share of Low-Income Buyers



(c) Falling Prices Over Product Age

(d) Falling Prices: Entrants vs. Incumbents

Notes. Each point plots the estimated coefficient from a regression of product-level outcomes on the cumulative years relative to initial year over the sample period. The x-axis measures years after initial sales records, and the y-axis shows the estimated change in the corresponding outcome relative to the initial year. All regressions include firm–product, firm–year, and product–year fixed effects, with standard errors clustered at the firm level. In Panel (a), the dependent variable is the average income quantile of a product's buyers, showing whether its customer base shifts toward lower-income consumers as it matures. In Panel (b), the dependent variable is the expenditure share of the bottom income tercile, reflecting the growing importance of poorer consumers in total spending. Panel (c) uses the annual price to track how average prices evolve after initial year. Panel (d) further illustrates that the decline in prices is driven concurrently by both new entrants and established firms. Solid dots represent point estimates, and vertical bars denote 95% confidence intervals based on robust standard errors.

Figure 2 shows how adoption of a module shifts across the income distribution as it ages. Each point is an event-time estimate of years since entry, controlling for firm–module, firm–year, and module–year fixed effects. The x-axis is years since the firm first sells in the module. The y-axis reports diffusion: the average buyer's income quantile in Panel (a) and the low-income expenditure share in Panel (b). Adoption clearly moves down the income distribution over time. In Panel (a), the average buyer's income rank falls by about four percentiles within twelve years of entry. Panel (b) shows a corresponding rise in the low-income expenditure share, indicating gradual penetration into poorer segments. A natural mechanism is falling prices as technology improves and competition intensifies (Jaravel, 2019; Bohr et al., 2023). We assess this channel by re-estimating the baseline specification with the module's annual price as the outcome.

As shown in Panel (c), average prices fall by about \$0.8 over the 12 years following entry, a decline of roughly 21 percent relative to the mean price of \$3.8. Panel (d) shows a similar downward price path in both the entrant and incumbent subsamples. Firms with no module entry during 2004–2016 are classified as incumbents, while all others are classified as entrants. This pattern suggests that the decline in prices reflects both within-firm adjustment and changes in the composition of sellers over the product life cycle, which in turn motivates the intensive- and extensive-margin decompositions in Facts 2 and 3. The diffusion pattern is consistent with a reservation-price mechanism (Simonovska, 2015; Bertoletti et al., 2018; Hu et al., 2025): lower-income households have lower willingness-to-pay thresholds, so declining prices naturally broaden product adoption among poorer consumers.[4]

Fact II: Innovation by Incumbents. Fact I shows that, as products age, prices fall and adoption shifts toward lower-income households. A natural mechanism is cost-reducing innovation by incumbent firms. We therefore examine how innovative activity varies with the income profile of a firm's customer base. Figure 3 plots firm-level innovation intensity, measured as the total number of patents per \$1,000 of consumer expenditure on the firm's products, separately for each year from 2004 to 2016. We split firms into two groups: *rich-market* firms, whose expenditure share from high-income consumers is above the median, and *poor-market* firms, which comprise the remainder.

The pattern is striking and stable across years. Firms serving high-income buyers innovate substantially more: in a typical year, rich-market firms file roughly four to six times as many patents per dollar of expenditure as poor-market firms. This concentration of innovation at the top of the income distribution is consistent with incumbents pushing the frontier in high-income segments, with subsequent diffusion as costs fall.

Fact III: Entry of New Competitors. Declining prices may reflect not only cost-reducing innovation by incumbents (Fact II) but also intensifying competition as new firms enter. To assess this market-structure channel, we estimate a module-year analogue of (1) in which the dependent variable is the number of active firms in module m and year t. The specification includes module fixed effects to absorb time-invariant differences across product categories and year fixed effects to absorb aggregate shocks; standard errors are clustered at the module level.

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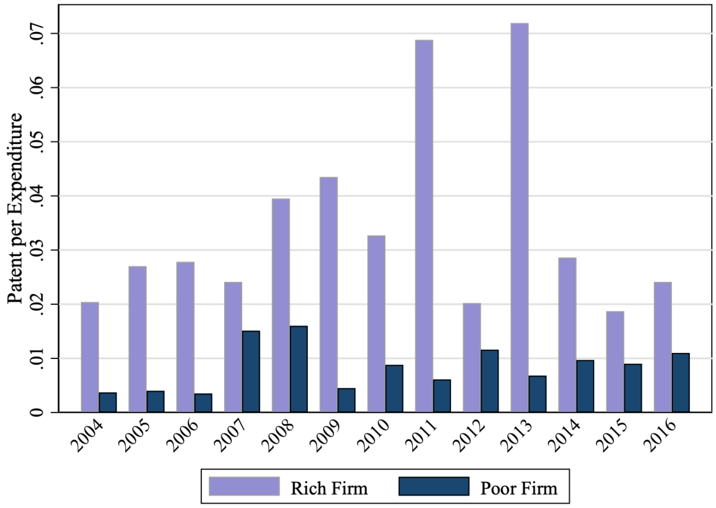
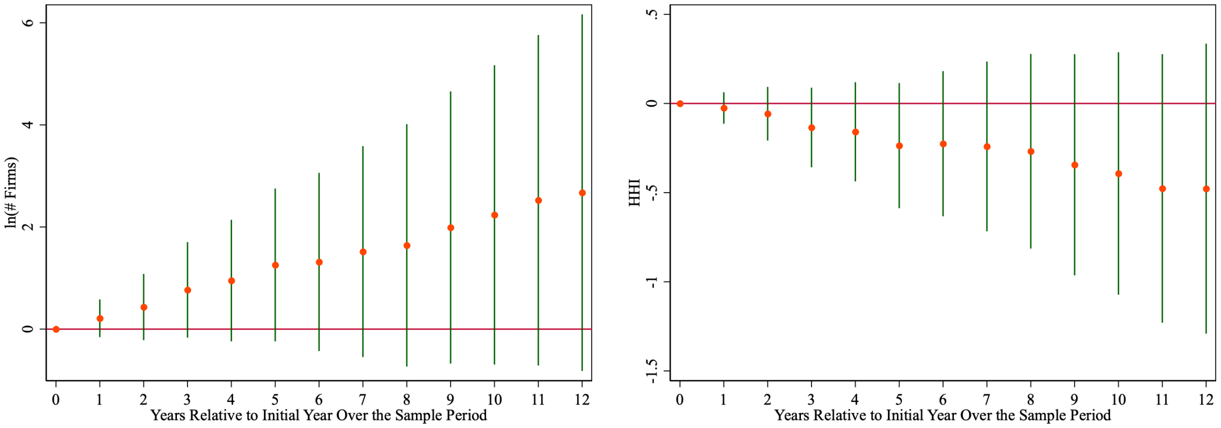
[4] Differences in demand elasticities and endogenous markups may also shape adoption patterns (Atkeson and Burstein, 2008), but their net effect on prices over the life cycle is theoretically ambiguous. Our reduced-form, module-level estimates therefore capture the average effect of these forces. The combination of steadily declining prices and rising participation by lower-income households suggests that, in this setting, cost reductions and competitive pressures outweigh any tendency for markups to rise.

Figure 3: Innovation Intensity by Customer Income Segment, 2004–2016



Notes. The figure plots, for each year, the mean innovation intensity of firms measured as the number of patents per \$1,000 of consumer expenditure on the firm's products. Firms are classified as *rich-market* if their expenditure share from high-income households is above the median, and as *poor-market* otherwise (high-income households are defined in the Nielsen data description). We exclude firm-year observations with zero expenditure from high-income households, for which the rich-market share is not defined.

Figure 4: Entry and Declining Concentration over the Product Life Cycle



(a) Number of Active Firms

(b) Market Concentration (HHI)

Notes. Each panel reports event-time estimates from regressions of module-year market-structure outcomes on years since the module's introduction. The x-axis measures years relative to the entry year (year 0). The y-axis shows the estimated change in the outcome relative to year 0. Panel (a) uses the log number of active firms in module m and year t. Panel (b) uses the Herfindahl–Hirschman Index, $HHI_{mt} = \sum_{f \in m} s_{ft}^2$, where s_{ft} is firm f's revenue share within module m in year t. All regressions include module fixed effects and year fixed effects. Dots denote point estimates and whiskers denote 95% confidence intervals based on standard errors clustered at the module level.

Figure 4 shows rapid entry following a module's introduction. Panel (a) indicates that within five years the number of active firms nearly doubles, pointing to substantial entry by new competitors. Panel (b) shows a parallel decline in concentration. The Herfindahl–Hirschman Index, $HHI_{mt} = \sum_{f \in m} s_{ft}^2$ falls steadily over event time, where s_{ft} is firm f's revenue share within module m in year t. Because the figure is designed to summarize the underlying pattern rather than to estimate a fully saturated specification with detailed fixed effects and controls, the year-by-year coefficients are naturally measured less precisely. Even so, the overall life-cycle pattern is clear: market participation broadens and concentration erodes as products mature, consistent with stronger competitive pressure that reinforces the price declines and downstream diffusion documented in Fact I.

Fact IV: Inequality and the Distribution of Innovation. If diffusion is powered by innovation and competition (Facts I–III), then changes in the income distribution should matter. We therefore study how income inequality affects firm innovation. A direct regression of patenting on inequality is hard to interpret because inequality and innovation may co-move: innovative firms can expand in particular markets, while changes in inequality can reshape the profitability of further innovation. To reduce this concern, we construct a Bartik-style, firm-specific exposure to inequality shocks using a predicted (and more plausibly exogenous) component of state-level inequality, denoted $Shock_{ft}^{ineq}$. Appendix A.7 provides details on measurement and the estimating equation.

$$Shock_{ft}^{ineq} = \sum_{s} \lambda_{sfto} \widehat{Gini}_{st}, \qquad \widehat{Gini}_{st} \equiv Gini_{s,2004} \times \frac{1}{N-1} \sum_{j \neq s} \frac{Gini_{jt}}{Gini_{j,2004}} \tag{2}$$

where f indexes firms, t indexes years, and s indexes U.S. states. The weights λ_{sft_0} are predetermined baseline expenditure shares: $\lambda_{sft_0} \equiv E_{sft_0} / \sum_f E_{sft_0}$, where E_{sft_0} is consumer expenditure on firm f in state s in the firm's entry year t_0. Following Boyce et al. (2016), $Gini_{st}$ is the state-year Gini index constructed from Nielsen; we also consider the Theil and Generalized Entropy (GE) indices.The predicted series $\widehat{Gini}_{st}$ follows Doerr et al. (2024): it interacts each state's initial inequality, $Gini_{s,2004}$, with the national evolution of inequality, measured by the average proportional change in the Gini across other states. Using $\widehat{Gini}_{st}$ rather than realized $Gini_{st}$ downweights state-specific movements that may be mechanically correlated with local firm outcomes while preserving the common shift in the income distribution. For the same reason, we also control for exposure to income levels, $Shock_{ft}^{inc}$, constructed analogously by replacing $\widehat{Gini}_{st}$ with a predicted state-year measure of mean household income.

Our outcome is firm-year patenting, measured as the number of patent applications filed by firm f in year t using PatentsView data from the U.S. Patent and Trademark Office. Because patent counts are highly skewed and many firm-years have zero applications, we use an inverse hyperbolic sine transformation of patent counts. This transformation retains zero-patent firm-years, avoids arbitrary adjustments such as adding a constant, and closely parallels a log specification when patent counts are larger (Doran et al., 2022). We control for firm characteristics that capture scale, market position, and scope: firm revenue, to distinguish innovation intensity from firm size; the firm's revenue share within its markets, as a proxy for market power; the number of modules supplied, to capture product scope; the number of states served, to capture geographic scope; and firm age, measured as years since first entry during 2004–2016. To ensure that the estimated inequality shift is mean-preserving rather than reflecting changes in income levels, we also control for firm exposure to income levels. Standard errors are clustered at the firm level throughout.

Table 1 reports the main results based on the Gini index. Greater exposure to inequality is associated with weaker firm innovation. In Panel A, Column (1) shows a negative and statistically significant effect on patenting. Columns (2)–(6) indicate that this result is robust across a range of alternative samples, including specifications that exclude entry years, restrict the sample to firms with meaningful exposure to both low- and high-income demand, focus only on patenting firms or large firms, and use a balanced panel. The effect also varies systematically across firms. Columns (7) and (8), which split firms by their dependence on low-income demand, show that the negative effect of inequality on patenting is concentrated among firms that sell more heavily to poorer households. For firms that are more oriented toward high-income consumers, the estimate is small and statistically insignificant. In economic terms, a one-standard-deviation increase in inequality exposure (8.83) reduces patenting by about 5.30%. Among firms with high exposure to low-income demand (12.41), the implied decline rises to 8.69%. Panel B further shows that these findings are not driven by low-value patents, as the estimates remain very similar when patent counts are weighted by claims. The same qualitative pattern also appears when inequality is measured using the Theil or GE indices; see Appendix Tables A.4 and A.5. Taken together, the evidence suggests that rising inequality not only reshapes the distribution of innovation across firms, but also depresses aggregate patenting, with the strongest effect falling on mass-market firms.

Summary of Motivational Facts. Our evidence reveals a systematic pattern in how consumer goods diffuse from higher-income to lower-income households. Over a module's life cycle, the expenditure share of poorer consumers rises, while both the average income of buyers and product prices decline (Fact I). Two forces appear to sustain this diffusion. First, demand from high-income consumers in the early stage of a product's life cycle is associated with stronger incumbent innovation, consistent with subsequent cost reductions over time (Fact II). Second, as the market expands, entry rises and concentration falls, pointing to stronger competition and further pressure on markups (Fact III). In turn, greater exposure to inequality is associated with weaker patenting, with the decline concentrated among firms that rely more heavily on low-income demand (Fact IV).

These patterns suggest that the distribution of income plays a central role in shaping how innovation begins and diffuses across consumers. When purchasing power is concentrated at the top, firms can cover fixed innovation costs by selling to affluent buyers first. High-end demand then finances experimentation and early cost reductions; as prices fall, the market expands and mass-market demand sustains further innovation and entry. Inequality thus shapes where innovation

Table 1: Income Inequality Exposure and Firm Innovation

	Baseline and Robustness						Heterogeneity	
Dep var. N_{ft}^{Pat}	(1) Full Sample	(2) No Entry Year	(3) Poor and Rich	(4) Innovative Firm	(5) Big Firm	(6) Surviving Frim	(7) Targeting Poor	(8) Targeting Rich
Panel A. Dependent variable: Patent applications								
$Shock_{ft}^{ineq}$	-0.006**	-0.007*	-0.006*	-0.006*	-0.006*	-0.006*	-0.007**	-0.002
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$Shock_{ft}^{inc}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.001
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Firm FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Observations	428,544	384,656	270,062	306,344	215,166	283,010	214,802	214,063
Adjusted R^2	0.672	0.681	0.678	0.655	0.682	0.680	0.678	0.689

	Baseline and Robustness						Heterogeneity	
Dep var. N_{ft}^{Pat}	(1) Full Sample	(2) No Entry Year	(3) Poor and Rich	(4) Innovative Firm	(5) Big Firm	(6) Surviving Frim	(7) Targeting Poor	(8) Targeting Rich
Panel B. Dependent variable: Claim-weighted patent applications								
$Shock_{ft}^{ineq}$	-0.010**	-0.011**	-0.010**	-0.010**	-0.010**	-0.010**	-0.011**	-0.007
	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$Shock_{ft}^{inc}$	-0.000	-0.000	-0.000	-0.001	-0.000	-0.000	-0.000	-0.002
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Firm FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Observations	428,544	384,656	270,062	306,344	215,166	283,010	214,802	214,063
Adjusted R^2	0.517	0.526	0.523	0.484	0.531	0.525	0.526	0.532

Notes. This table reports firm-year regressions of innovation on exposure to income-inequality shocks. In Panel A, the dependent variable is IHS($Patents_{ft}$), where $Patents_{ft}$ is the number of patent applications filed by firm f in year t (PatentsView/USPTO). In Panel B, the dependent variable is IHS($Claim\text{-}weighted\ Patents_{ft}$), where patents are weighted by the number of claims. The main regressor, $Shock_{ft}^{ineq}$, is the firm-specific inequality exposure in equation (2): a weighted average of predicted state-year inequality, with weights given by the firm's baseline expenditure shares across states in its entry year. We also control for $Shock_{ft}^{inc}$, constructed analogously by replacing predicted inequality with a predicted state-year measure of mean household income; this control separates changes in dispersion from shifts in income levels. Columns (1)–(6) report baseline and robustness samples: (2) excludes each firm's entry year; (3) restricts to firms with positive exposure to both low- and high-income consumers; (4) restricts to firms that patent at least once during 2004–2016; (5) restricts to firms with above-median revenues; and (6) restricts to firms observed in all sample years (balanced panel). Columns (7)–(8) split firms by sales orientation: *Targeting Poor* (*Targeting Rich*) indicates firms with above-median (below-median) expenditure shares from low-income consumers, computed over the sample period. All specifications include firm and year fixed effects. Controls include $\ln(1 + Revenues_{ft})$, the firm's revenue share (market-power proxy), $\ln(1 + \#Modules_{ft})$, $\ln(1 + \#States_{ft})$, and firm age. Standard errors (in parentheses) are clustered at the firm level. For readability, $Shock_{ft}^{ineq}$ and $Shock_{ft}^{inc}$ are multiplied by 1,000. ***, **, and * denote significance at the 1%, 5%, and 10% levels, respectively.

starts and how quickly its gains diffuse. To discipline these empirical regularities and assess how the income distribution shapes firm dynamics and long-run growth, the model below formalizes the underlying mechanism.

3 Model

In this section, we build a dynamic general-equilibrium model that links heterogeneous households on a quality ladder to innovating, heterogeneous firms with endogenous product variety.

3.1 Households

The economy is populated by a continuum of households with identical preferences but heterogeneous incomes. Each household is endowed with labor l, drawn from a continuous and differentiable distribution $\mathcal{G}(l)$ over $[\underline{l}, \infty)$ with mean normalized to one, $\mathbb{E}[l] = 1$. This distribution also represents the income distribution in our model. Preferences are defined over a continuum of vertically differentiated final goods, capturing a hierarchical structure of needs.

Following Matsuyama (2002) and Foellmi and Zweimüller (2006), we assume that a household with endowment l derives utility from effective consumption:

$$U_l(0) = \int_0^\infty e^{-\rho t} \frac{C_l(t)^{1-\sigma}}{1-\sigma} dt$$

where $\rho > 0$ is the discount rate, $1/\sigma$ is the intertemporal elasticity, and

$$C_l(t) = \int_0^{N(t)} i^{-\gamma} C_l(i,t) di + \min_{\{i \in (0, N_t]\}} \{(C_l(i,t))\} \cdot X_l(t), \tag{3}$$

where $C_l(i,t) \in \{0,1\}$ denotes a binary decision to consume variety i, $N(t)$ is the measure of available differentiated goods, and $X_l(t)$ is a continuously consumed luxury service. Assume that $\gamma \in [0,1)$. The index $i \in (0, N(t)]$ orders goods by consumption priority: lower-i goods are more basic as they carry a higher utility weight.[5] The luxury service $X_l(t)$ yields utility only if all manufactured varieties are consumed, i.e., if $\min_{i \in (0, N(t)]} C_l(i,t) = 1$[6].

Let $P(i,t)$ denote the price of manufactured variety i and $P^X(t)$ the price of the luxury service $X_l(t)$. Labor is supplied inelastically at the prevailing wage $W(t)$. Following Foellmi and Zweimüller (2006), we assume further that each household has the same income composition (identical labor

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[5] Because consumption of each manufactured variety is binary, the conventional classification of goods into "luxuries" and "necessities" based on income elasticities is not well defined: the income elasticity of demand is either zero or infinite. Following Matsuyama (2002), we instead adopt an expenditure-share–based definition. A good is defined as a necessity if its expenditure share rises when household income falls, and as a luxury if its expenditure share rises when household income increases.

[6] Our specification of the luxury service is similar to the treatment of leisure in Matsuyama (2002). In his model, sufficiently wealthy households reduce labor supply once they have consumed all discrete goods, deriving utility from leisure at the margin. In contrast, we assume that such households allocate resources to a continuously consumed luxury service. The key difference is that inequality affects aggregate labor supply in Matsuyama (2002), while aggregate labor supply remains constant in our setting.

and profit shares) and has no initial saving at $t = 0$. For a household with labor endowment l, the present-value budget constraint is given by

$$\int_0^\infty e^{-R(t,0)} \left[\int_0^{N(t)} P(i,t) C_l(i,t) di + P^X(t) X_l(t) \right] dt \leq \int_0^\infty W(t) l \cdot e^{-R(t,0)} dt + \mathbb{V}(0) l,$$

where $R(t,0) \equiv \int_0^t r(s)\, ds$ is the cumulative interest factor and $\mathbb{V}(0)$ is the aggregate market value of all firms at $t = 0$ defined in equation (B.23) in the appendix.[7] For tractability, we assume that each household owns l shares of all firms as the expected labor endowment l is equal to one. Thus, a household with a larger l has a higher income level.

Given the binary nature of consumption choices, households follow a reservation-price strategy. For each differentiated good i, a household with labor endowment l purchases the good if and only if its market price does not exceed its reservation price. Solving the household's utility-maximization problem yields the consumption rule:

$$C_l(i,t) = \begin{cases} 1, & P(i,t) \leq \kappa_l(i,t), \\ 0, & P(i,t) > \kappa_l(i,t), \end{cases} \quad \text{with} \quad \kappa_l(i,t) = \frac{i^{-\gamma} e^{-\rho t}}{\mu_l C_l(t)^\sigma} \cdot e^{R(t,0)}, \tag{4}$$

where $\kappa_l(i,t)$ denotes the reservation price of good i for a household of type l at time t, μ_l represents the marginal utility of wealth (the Lagrange multiplier associated with the household's budget constraint). In the balanced growth equilibrium (BGE) formally defined later, $P(i,t)$ is increasing in i and $\kappa_l(i,t)$ is decreasing in i, which leads to a a hierarchical consumption structure: goods are purchased in an increasing order of i, and only households with sufficiently high income consume higher i varieties. Furthermore, only households whose income is high enough to consume all $N(t)$ varieties allocate their remaining expenditure to luxury services. Lemma B.7 in Appendix B.3.1 characterizes this consumption pattern.

3.2 Production

On the supply side, the economy consists of three types of firms: competitive producers of final goods, monopolistically competitive producers of intermediate goods, and competitive luxury service providers.

Final Goods Producers. While much of the related literature typically assumes a single homogeneous final good, our framework introduces a continuum of final goods $i \in (0, N(t)]$, each linked to its own range of intermediates. This structure captures heterogeneity in sectoral specialization and allows the scope of product variety to evolve endogenously over time. For each final-good variety i, a unit mass of identical perfectly competitive firms produces output using a measure $F(i,t)$ of specialized intermediate inputs supplied by firms indexed by $j \in [0, F(i,t)]$. The representative

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[7] The present-value budget constraint implies that households are allowed to save at the interest rate $r(t)$. However, we will show in the proof of Lemma B.8 in the Appendix that, along the BGP, net savings are zero for every households.

final-good producer operates with the decreasing-returns-to-scale technology:

$$Y(i,t) = \frac{1}{1-\beta}\int_0^{F(i,t)} M_i(j,t)^{1-\beta}dj,$$

where $M_i(j,t)$ denotes the quantity of intermediate input j used in the production of good i, and $\beta \in (0,1)$ governs the elasticity of substitution across intermediates[8]. A larger $F(i,t)$ reflects a broader range of specialized suppliers within sector i, raising productivity through the variety-expansion channel.

The representative final-good producer chooses the set of inputs $M_i(j,t)$ to maximize profits, taking prices as given:

$$\Pi(i,t) \equiv \max_{\{M_i(j,t)\}} \left\{ \frac{P(i,t)}{1-\beta}\int_0^{F(i,t)} M_i(j,t)^{1-\beta}dj - \int_0^{F(i,t)} Q_i(j,t)M_i(j,t)dj \right\}, \tag{5}$$

where $P(i,t)$ is the price of final good i, and $Q_i(j,t)$ is the price of intermediate good j. The first-order condition of this problem yields the demand for intermediate inputs:

$$M_i(j,t) = \left[\frac{P(i,t)}{Q_i(j,t)}\right]^{\frac{1}{\beta}}. \tag{6}$$

We define sector i as the collection of intermediate producers whose outputs are exclusively used in the production of final good i. This formulation highlights two structural forces that shape modern production networks: the deepening of intra-sectoral specialization and the endogenous expansion of input variety through innovation and market entry.

Intermediate Goods Producers. Within each sector i, a continuum of $F(i,t)$ monopolistically competitive firms produces differentiated intermediate goods. Each firm operates a linear production technology:

$$M_i(j,t) = \left(\tilde{A}_i(j,t)^{\theta}\bar{A}(t)^{1-\theta}\right) L_i(j,t), \tag{7}$$

where $L_i(j,t)$ denotes labor employed by firm j in sector i. Firm productivity has two components: an idiosyncratic term $\tilde{A}_i(j,t)$, and an aggregate productivity term $\bar{A}(t)$, which will be formally defined below. The Cobb–Douglas combination of these components captures the interaction between firm-specific efficiency and economy-wide technological progress, with $\theta \in (0,1)$ determining their relative importance. Each intermediate producer chooses its labor hired to maximize profits:

$$\max_{L_i(j,t)} \left\{Q_i(j,t)M_i(j,t) - W(t)L_i(j,t)\right\}, \tag{8}$$

subject to equation (6) and equation (7), where $W(t)$ is the wage rate.

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[8] Here we assume a decreasing-returns-to-scale production technology. In Appendix C.2, we consider an alternative specification with constant returns to scale, and show that all the theoretical results continue to hold.

The first-order condition of (8) yields firm profits:

$$\Pi_i(j,t) = \beta(1-\beta)^{\frac{1-\beta}{\beta}} \left[\frac{P(i,t)^{\frac{1}{1-\beta}}}{W(t)} \bar{A}(t)^{(1-\theta)} \tilde{A}_i(j,t)^{\theta} \right]^{\frac{1-\beta}{\beta}}. \tag{9}$$

equation (9) shows that profits rise with both the sectoral price $P(i,t)$ and the firm's own productivity $\tilde{A}_i(j,t)$. To aggregate across firms, we define sector-level productivity as

$$B(i,t) \equiv \bar{A}(t)^{(1-\theta)\frac{1-\beta}{\beta}} \int_0^{F(i,t)} \tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}} dj \tag{10}$$

The term $B(i,t)$ summarizes the effective productivity of sector i, which increases with both firm-level productivity and the number of intermediate producers. The positive relationship between $B(i,t)$ and $F(i,t)$ arises from decreasing marginal returns to each intermediate input in final-goods production, implying that a richer network of suppliers raises sectoral efficiency. Given this representation, sectoral output supply and labor demand can be expressed as:

$$Y(i,t) = (1-\beta)^{\frac{1-2\beta}{\beta}} \left[\frac{W(t)}{P(i,t)} \right]^{-\frac{1-\beta}{\beta}} B(i,t), \tag{11}$$

$$L(i,t) = \left[\frac{W(t)}{(1-\beta)P(i,t)} \right]^{-\frac{1}{\beta}} B(i,t), \tag{12}$$

both are proportional to sectoral productivity $B(i,t)$. The novel feature here lies in the structure of the relative price $P(i,t)$. Rather than serving as a mere equilibrium outcome, $P(i,t)$ encapsulates consumers' willingness to pay along the hierarchy of needs, linking demand heterogeneity directly to production outcomes.

Following the standard expanding-variety framework of Romer (1990), we interpret $N(t)$—the number of available final-good varieties—as a proxy for aggregate productivity, and simply set $\bar{A}(t) = N(t)$. This formulation captures the idea that technological progress operates primarily through the expansion of product varieties. Alternative measures—such as the economy-wide average firm productivity would yield similar implications. The analysis is reported in the Appendix.

Luxury Service Providers. Finally, we introduce the production of luxury services. These services are supplied by representative competitive firms operating a linear technology:

$$X(t) = \bar{A}(t)^{1-\gamma} L^X(t),\ \gamma \in (0,1),$$

where $L^X(t)$ denotes labor employed in the luxury service sector. The exponent $1-\gamma$ captures the degree to which productivity improvements in goods-producing sectors spill over to luxury services. This specification ensures the existence of BGP with a stationary relative price of luxury services $P^X(t)$ and keeps the utility contributions from goods and services comparable along the growth path. Firms in this sector operate under perfect competition and do not engage in innovation on their own. Because technological progress in the luxury service sector is not our focus, we simply assume productivity growth in luxury service sector arises entirely through spillovers from the goods-producing sectors, as aggregate productivity $\bar{A}(t)$ diffuses throughout the economy.

3.3 Innovation

Innovation in the economy takes three distinct forms. First, incumbent intermediate-good producers engage in R&D to improve their own productivity—what we refer to as *incumbents' innovation*. Second, potential entrants invest in R&D to introduce new intermediate varieties, a process we label *firm entry*. Finally, a separate group of inventors develops entirely new final goods, thereby creating new consumption categories and expanding the economy's sectoral frontier—a process we refer to as *new-good invention*.

Incumbents' Innovation We start with incumbents' R&D decisions. An incumbent that invests in R&D faces a Poisson arrival rate of successful innovation, denoted by z. Each success raises the firm's productivity by a factor $\lambda > 1$. For firm j in sector i, the innovation arrival rate is

$$z_i(j,t) = \psi \left(\frac{\tilde{A}_i(j,t)}{\bar{A}(t)} \right)^{-\iota} \left(\bar{A}(t) L_i^I(j,t) \right)^{\alpha}, \tag{13}$$

where $L_i^I(j,t)$ is labor allocated to R&D. The technology is concave in R&D effort, with $\alpha \in (0,1)$ governing the elasticity of innovation with respect to R&D labor and $\psi > 0$ indexing baseline R&D efficiency. As in standard endogenous-growth models, aggregate productivity $\bar{A}(t)$ enters R&D as an external knowledge stock, so that a given amount of research labor is more productive in a more advanced economy. The term $\left(\tilde{A}_i(j,t)/\bar{A}(t) \right)^{-\iota}$ captures diminishing returns to pushing the frontier: conditional on research effort, relatively more productive firms face a lower marginal effectiveness of R&D, which prevents systematically faster growth for already-leading incumbents. We impose the following parameter restriction:

Assumption 1 *The parameter ι satisfies $\iota = \alpha\theta\frac{1-\beta}{\beta}$.*

Assumption 1 imposes a standard scaling restriction that keeps the incumbent's R&D problem homogeneous (see, e.g., Akcigit and Kerr (2018); Peters (2020)). With this restriction, a firm's productivity affects incentives only through a normalization, so $\tilde{A}_i(j,t)$ need not be tracked as an additional state variable. The optimal R&D policy can therefore be written in terms of normalized sectoral and aggregate objects, which delivers a tractable characterization of innovation behavior.

Assume that all incumbent firms face an exogenous exit risk, modeled as a Poisson process with constant rate δ. Then the value function of firm j with productivity $\tilde{A}_i(j,t)$, denoted $\tilde{V}_i(\tilde{A}_i(j,t),t)$, satisfies the following Hamilton–Jacobi–Bellman (HJB) equation along the BGP:

$$\begin{aligned} r\tilde{V}_i(\tilde{A}_i(j,t),t) - \dot{\tilde{V}}_i(\tilde{A}_i(j,t),t) \quad = \quad & \max_{L_i^I(j,t)} \Pi_i(j,t) - W(t) \cdot L_i^I(j,t) \\ & + z_i(j,t) \cdot \left[\tilde{V}_i(\lambda \tilde{A}_i(j,t),t) - \tilde{V}_i(\tilde{A}_i(j,t),t)\right] \\ & - \delta \tilde{V}_i(\tilde{A}_i(j,t),t), \end{aligned} \tag{14}$$

where r is the equilibrium interest rate along the BGP. The first line on the right-hand side represents the firm's operating profit net of R&D costs, while the second and the third capture the expected gain from successful innovation and the loss from exogenous exit, respectively. In Lemma B.1 of Appendix B.2.2, we provide a characterization of the value function

$$\tilde{V}_i(A,t) = A^{\theta \frac{1-\beta}{\beta}} V_i(t), \tag{15}$$

and show that the R&D intensity $z_i(j,t)$ is identical across firms j in the same sector i and depends solely on sector-level fundamentals. We simply write it as $z_i(t)$. This property substantially simplifies the characterization of equilibrium innovation dynamics.

Firm Entry As a second type of innovation, potential entrepreneurs in each sector i conduct R&D aimed at creating new firms. Entry is stochastic and follows a Poisson process, with the arrival rate of successful innovation given by

$$e(i,t) = \psi_e \bar{A}(t) L_i^E(t), \tag{16}$$

where $L_i^E(t)$ denotes the amount of labor devoted to R&D for new firm creation in sector i, and $\psi_e > 0$ measures the efficiency of the entry process. Upon a successful innovation, a new firm is created and assigned a productivity level A drawn from the existing distribution of incumbent firms within the same sector. The optimization problem of potential entrants in sector i is given by

$$\max_{L_i^E(t)} \left\{ e(i,t) \cdot \mathbb{E}_t[A^{\theta \frac{1-\beta}{\beta}} V_i(t)] - W(t) \cdot L_i^E(t) \right\}, \tag{17}$$

where $\mathbb{E}_t[A^{\theta \frac{1-\beta}{\beta}} V_i(t)]$ represents the expected value of a newly created firm conditional on successful entry. The number of firms in sector i evolves according to

$$\dot{F}(i,t) = e(i,t) - \delta F(i,t), \tag{18}$$

where $e(i,t)$ captures the inflow of new entrants and $\delta F(i,t)$ represents the exit of incumbent firms due to exogenous shocks. More technical details on firm entry are provided in Appendix B.2.

Given this structure, sectoral productivity is determined jointly by incumbent innovation and firm dynamics. As defined in equation (10), sectoral productivity $B(i,t)$ depends on aggregate productivity $\bar{A}(t)$, the number of firms $F(i,t)$, and firm-specific productivity $\tilde{A}_i(j,t)$. As shown in

Appendix B.3.3, the evolution of $B(i,t)$ follows

$$\frac{\dot{B}(i,t)}{B(i,t)} = \underbrace{(1-\theta)\frac{1-\beta}{\beta}\frac{\dot{\bar{A}}(t)}{\bar{A}(t)}}_{\text{technology spillovers}} + \underbrace{z_i(t)\cdot\left(\lambda^{\theta\frac{1-\beta}{\beta}}-1\right)}_{\text{incumbents' innovation}} + \underbrace{\frac{e(i,t)}{F(i,t)}-\delta}_{\text{firm dynamics}}\,. \tag{19}$$

equation (19) indicates that sectoral productivity growth is shaped by three key forces: aggregate technological spillovers, innovation by incumbent firms, and the entry of new firms. While new entrants do not immediately raise average productivity—since their productivity is drawn from the existing distribution—they expand the range of intermediate inputs available for production. This expansion along the extensive margin increases the diversity of inputs, improves resource allocation within the sector, and ultimately contributes to higher overall productivity growth.

New-Good Invention New-good invention creates entirely new categories of final consumption goods, expanding the consumption frontier by generating new sectors. It is carried out by a continuum of inventors embedded in households and distributed according to the income distribution $\mathcal{G}(l)$, so that their consumption is fully internalized in aggregate household consumption. Inventors undertake R&D by hiring labor, and successful invention follows a Poisson process with an arrival rate linear in invention labor and aggregate productivity. The law of motion for the number of final goods is given by

$$\dot{N}(t) = \phi\bar{A}(t)L^N(t), \tag{20}$$

where $L^N(t)$ denotes the total labor input devoted to invention activities, and $\phi > 0$ measures the efficiency of the invention process. Each successful invention gives rise to a new sector producing a distinct final good. Upon invention, each inventor is endowed with ownership of a unit mass of final-goods firms in the newly established sector. The initial intermediate entrants in that sector begin with a baseline productivity level equal to a fraction ζ of aggregate productivity $\bar{A}(t)$, i.e. $\tilde{A}_{N(t)}(j,t) = \zeta\bar{A}(t)$ for all $j \in [0, F(N(t),t)]$. In the equilibrium, ζ is assumed to be relatively small to reflect that newly created sectors typically start with lower productivity. The precise restriction is given in equation (B.68).

The inventor's optimization problem can therefore be written as

$$\max_{L^N(t)}\{V^F(N(t),t)\dot{N}(t) - W(t)L^N(t)\}, \tag{21}$$

where $V^F(N(t),t)$ denotes the value of a final-goods firm in a newly established sector, corresponding to the present discounted value of all future profits as defined in equation (5). This formulation highlights that inventors' incentives to create new consumption goods depend on the profitability of final-goods producers, which in turn reflects household demand for newly invented varieties. Because demand declines with the index $N(t)$, aggregate income constrains the market size for high-$N(t)$ goods. Consequently, inventors have limited incentive to develop excessively luxurious varieties, as the expected firm value $V^F(N(t),t)$ becomes too low to justify invention costs. As aggregate productivity $\bar{A}(t)$ rises, however, higher household income expands effective demand, making the invention of more luxurious goods profitable.

Given $N(t) = \bar{A}(t)$, equation (20) implies that

$$g_n \equiv \frac{\dot{N(t)}}{N(t)} = \phi L^N(t).$$

As in Romer (1990), the engine of economic growth is driven by the growth of $N(t)$, which is determined by the labor $L^N(t)$ for the final product innovation. As $L^N(t)$ is endogenously determined by the labor market clearing condition, a key feature of our model is that hierarchical consumer demand and labor allocation in other sectors of the economy play a central role in economic growth.

4 Balanced Growth Equilibrium

In this section, we characterize the balanced growth equilibrium (BGE) and study properties of such an equilibrium.

4.1 Characterizing the BGE

We characterize the equilibrium along a BGP. When doing so, we suppress the time index t whenever no confusion arises. Define aggregate final consumption index as

$$C(t) = \int_{\underline{l}}^{\infty} C_l(t) d\mathcal{G}(l).$$

We normalize the price index P^C of this aggregate final consumption bundle to 1. Define aggregate output as

$$Y(t) = \int_0^{N(t)} P(i,t)Y(i,t)di + X(t)P^X(t),$$

where X denotes aggregate luxury services and satisfies

$$X(t) = \int_{\underline{l}}^{\infty} X_l(t) d\mathcal{G}(l)$$

in equilibrium. In the proof of Proposition 1, we show that $Y(t) = C(t)$ holds in a BGE.

Definition 1 *A BGE consists of paths of prices* $\{W, r, P^X, \{P(i)\}_i, \{Q_i(j)\}_{i,j}\}$*, quantities* $\{N, \{F(i)\}_i, \{Y(i)\}_i, \{M_i(j)\}_{i,j}, \{C_l(i)\}_{l,i}, \{X_l\}_l, X, C, Y\}$*, and labor choices* $\{\{L_i(j)\}_{i,j}, \{L_i^I(j)\}_{i,j}, \{L_i^E\}_i, L^N, L^X\}$ *such that: (i) households maximize lifetime utility subject to their budget constraints taking interest rates and prices as given; (ii) competitive final-good producers in each sector* i *chooses intermediate goods to maximize their profits; (ii) monopolistically competitive intermediate goods firms for each sector* i *chooses labor to maximize profits, taking downward-sloping demand curve as given; (iii) incumbent intermediate goods firms hire labor to conduct innovations to maximize firm value; (iv) entrants and new-good innovators choose labor to solve problems (17) and (21) taking wages* *as given; (v) the mass of intermediate goods firms $F(i)$ satisfies equation (18); (vi) competitive luxury services providers choose labor to maximize profits. (vii) markets for final goods, intermediate goods, luxury services, and labor all clear; and (viii) along the BGP, r and P^X are constant, $\{C, X, Y, W, N\}$, $\{C_l\}_l$, and $\{X_l\}_l$ grow at constant rates.*

In Appendix B, we show that, along the BGP, Y, C, X, C_l, and W all grow at the same rate $(1-\gamma)g_n$, and $X_l = 0$ for households with $l < \bar{l}$, but X_l grows at rate $(1-\gamma)g_n$ for households with $l \geq \bar{l}$. But at any point in time t^*, as a new sector $N(t^*)$ is created, the number of firms in that sector, $F(N(t^*), t)$, changes with t due to entry and exit and approaches a constant in the long run. As other new products are being invented over time, product $N(t^*)$ moves down the product ladder. As aggregate labor supply is fixed at 1 and sectors expand at rate g_n, employment and entrant R&D labor input in each sector decline at rate g_n.

A convenient way to express the BGP in stationary form is to normalize the product space from $(0, N(t)]$ to $(0, 1]$. Accordingly, define the normalized product index

$$\tilde{i}(t) \equiv \frac{i}{N(t)}.$$

This index orders goods by their relative position along the product ladder and plays a central role in characterizing sectoral outcomes along the BGP. In Lemma B.10, we show that two sectors with the same $\tilde{i}$ at different points in time will share identical (possibly detrended) values for output, innovation rates, firm numbers, and other sector-specific variables.

Based on this property, each sectoral variable can be rewritten as the product of a common trend and a stationary profile in $\tilde{i}$. Throughout, detrended variables are written in lowercase. As an illustration, let $V_i(t)$ denote the productivity-adjusted firm value defined in equation (15), and define its detrended counterpart as $v(\tilde{i}) \equiv e^{-g_v t} V_i(t)$, where g_v is the constant growth rate of $V_i(t)$. Once $v(\tilde{i})$ is determined, all sectoral supply-side variables—including innovation intensity and productivity—are uniquely pinned down, so that $v(\tilde{i})$ summarizes the relevant supply-side information.

The same approach can be applied to the demand side. Define $\mathcal{L}(\tilde{i}(t))$ as the lowest labor endowment level among households that purchase product $\tilde{i}$ at time t:

$$\mathcal{L}(\tilde{i}(t)) = \inf\{l : C_l(i, t) = 1\}.$$

The household with $l = \mathcal{L}(\tilde{i}(t))$ is the marginal buyer such that any household with $l' > l$ will also buy product $\tilde{i}$ at time t. The function $\mathcal{L}(\tilde{i})$ provides a compact summary of the demand for good $\tilde{i}$. The detailed characterization of $\mathcal{L}(\tilde{i})$ can be found in Appendix B.3.1.

In Proposition 1, we show that the BGE can be mathematically characterized by a joint system for $v(\tilde{i})$ and $\mathcal{L}(\tilde{i})$, together with a small number of endogenous scalars including the growth rate g_n of N.

Proposition 1 *The BGE can be characterized by two functions, $\{v(\tilde{i}), \mathcal{L}(\tilde{i})\}$, $\tilde{i} \in (0, 1]$, and four* *endogenous variables,* $\{w, \eta, y, g_n\}$*, which are jointly determined by the following system:*

(i) The function $v(\tilde{i})$ *solves*

$$v'(\tilde{i}) = \theta\frac{1-\beta}{\beta\tilde{i}} \cdot \left[\left(\frac{v(\tilde{i})}{h(g_n)}\right)^{\frac{\alpha}{1-\alpha}} - 1\right] v(\tilde{i}), \tag{22}$$

subject to the boundary condition

$$v(1) = \frac{w}{\psi_e \zeta^{\theta\frac{1-\beta}{\beta}}}.$$

Here, $h(\cdot)$ *is given by*

$$h(g_n) = \frac{w\left(\theta\frac{1-\beta}{\beta}\right)^{\frac{1-\alpha}{\alpha}}}{\left[\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right)\psi\right]^{\frac{1}{\alpha}}\alpha} g_n^{\frac{1-\alpha}{\alpha}}.$$

(ii) There exists a constant $\eta > 0$ *denoting the share of product varieties universally consumed across all households. If* $\tilde{i} \geq \eta$*,* $\mathcal{L}(\tilde{i})$ *satisfies*

$$\mathcal{L}'(\tilde{i}) = \frac{w^{1-\beta}}{y\beta^{\beta}(1-\beta)^{1-\beta}} \left\{\left[\rho + \delta + \left((\sigma-1)(1-\gamma) + 1 - \theta\frac{1-\beta}{\beta}\right) g_n\right] v(\tilde{i}) + w\left[\frac{w}{\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right) v(\tilde{i}) \cdot \psi\alpha}\right]^{\frac{1}{\alpha-1}}\right\}^{\beta}, \tag{23}$$

subject to the boundary conditions $\mathcal{L}(\eta) = \underline{l}$*. For* $\tilde{i} \in (0, \eta)$*,* $\mathcal{L}(\tilde{i}) = \underline{l}$*.*

(iii) The four variables are determined by four conditions detailed in Appendix B.4: w *from the labor market clearing condition;* η *from the resource constraint;* y *from the price normalization;* g_n *from the entry condition of final producers.*

The proof is shown in Appendix B.4. This proposition characterizes the equilibrium through two ODEs. Equation (22) is derived from the firm entry condition together with the dynamics of sector-level productivity. Its boundary condition reflects the assumption that the productivity of any newly established sector equals a fraction ζ of the aggregate productivity. The function $h(\cdot)$ corresponds to $\lim_{\tilde{i}\to 0} v(\tilde{i})$ for a given growth rate g_n. Equation (23), in turn, is obtained from the incumbent firms' HJB equation combined with the demand curve summarized by $\mathcal{L}(\tilde{i})$. The boundary condition shows that the poorest household with endowment $\underline{l}$ buys all products $\tilde{i} \in (0, \eta]$. This means that sectors in $(0, \eta]$ produce basic products that are consumed by all consumers. Figure B.1 in Appendix B.1 illustrates our model structure and equilibrium feature.

4.2 Sectoral Dynamics along the BGP

Beyond the technical characterization in Proposition 1, our model admits tractable qualitative properties of sectoral dynamics. In particular, as one moves up the product ladder (i.e., to higher $\tilde{i}$), the equilibrium exhibits monotone patterns in productivity, prices, firm value, and innovation incentives. As shown in Appendix B.3.3, we define detrended variables:

$$z(\tilde{i}) = z_i(t),\ b(\tilde{i}) = e^{-\frac{1-\beta}{\beta} g_n t} B(i,t),\ p(\tilde{i}) = e^{\gamma g_n t} P(i,t).$$

Proposition 2 presents the result with its proof given in Appendix B.5.

Proposition 2 *Along the BGP, for* $\tilde{i} \in (0,1]$*, detrended sectoral productivity* $b(\tilde{i})$ *is strictly decreasing in* $\tilde{i}$*, whereas the detrended price* $p(\tilde{i})$*, productivity-adjusted firm value* $v(\tilde{i})$*, and innovation intensity* $z(\tilde{i})$ *are strictly increasing. Moreover,*

$$p(\tilde{i}) = y\mathcal{L}'(\tilde{i}) \ \ \textit{for} \ \ \tilde{i} \in [\eta, 1). \tag{24}$$

Proposition 2 has four implications. First, the detrended productivity $b(\tilde{i})$ is lower in more luxurious sectors (higher $\tilde{i}$). Equivalently, within any given sector i, the productivity $B(i,t)$ rises over time as the sector matures, reflecting both incumbent innovation and entry-induced reallocation.

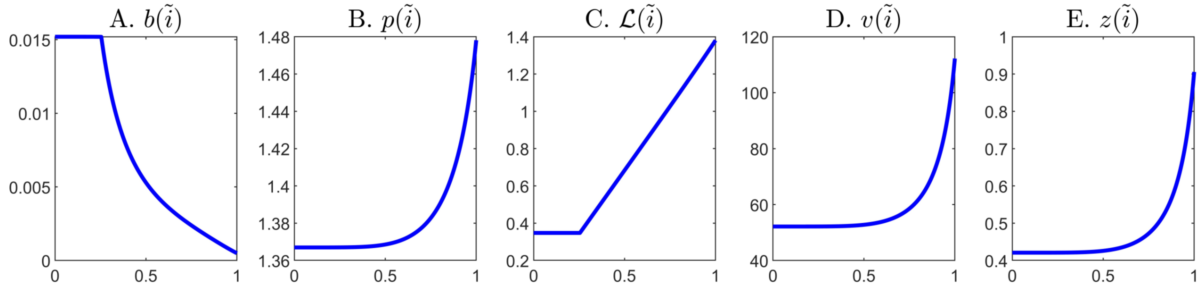
Second, the detrended price is higher for a more luxurious product. Intuitively, new sectors for more luxurious products begin with lower productivity and therefore higher marginal costs. With constant markups for both intermediate- and final-good producers, these higher costs translate one-for-one into higher prices. This result implies that, for any fixed sector i, the product price $P(i,t)$ declines over time at an asymptotic rate γg_n. As product ages, incumbents conduct innovations to improve productivity so that the price falls over time. The trend growth reflects aggregate productivity growth. Equation (24) shows that, in detrended terms, the price of product $\tilde{i} \in (0, \eta]$ is equal to the total income that the marginal buyer is willing to spend for the product with an additional level of luxury.

Third, the productivity-adjusted firm value $v(\tilde{i})$ increases with $\tilde{i}$. This object measures the value of a marginal improvement in productivity. Because luxury sectors sustain higher prices, an additional unit of output is worth more, raising the payoff to productivity gains and hence $v(\tilde{i})$.

Finally, the innovation intensity $z(\tilde{i})$ rises with the luxury level. Intuitively, a higher productivity-adjusted value strengthens the incentive to invest in R&D, increasing the Poisson arrival rate of innovation. These monotone sectoral patterns yield a set of testable predictions, and they line up with the empirical regularities documented in Section 2.2.

Figure 5 illustrates Proposition 2 based on the calibrated parameter values in Section 5. Notice that all lines are almost flat when $\tilde{i}$ is below $\eta = 0.254$ because the market size does not expand as all households consumes products in $(0, \eta]$. In Panel C, $\mathcal{L}(\tilde{i})$ is exactly flat at $\underline{l}$ on $(0, \eta)$ and is increasing and convex on $[\eta, 1)$.

Figure 5: Sectoral Dynamics



Notes. The figure illustrates several detrended sectoral variables under the parameter listed in Table 2, including: detrended productivity $b(\tilde{i})$ in Panel A, detrended product prices $p(\tilde{i})$ in Panel B, the marginal buyer $\mathcal{L}(\tilde{i})$ in Panel C, productivity-adjusted firm value $v(\tilde{i})$ in Panel D, and innovation intensity $z(\tilde{i})$ in Panel E.

4.3 Innovation and Growth

Rising inequality changes not only who consumes, but also what firms find profitable to improve. In our model, it shifts the composition of demand along the product ladder and thereby tilts R&D incentives across sectors. To make this transparent, we consider a permanent change in the income distribution $\mathcal{G}$ and hold all other parameter values fixed.

Proposition 3 *Consider two income distributions such that* $g_n^1 > g_n^2$ *in the BGE. Then the logarithmic change in sector-level innovation intensity,* $\log z^1(\tilde{i}) - \log z^2(\tilde{i})$*, is positive for all* $\tilde{i}$ *and strictly decreasing in* $\tilde{i}$*.*

Proposition 3 delivers two key implications, and its proof is given in Appendix B.6. First, aggregate growth and innovation move together: any change in income distribution that lowers (raises) the long-run growth rate also lowers (raises) innovation intensity in every sector. Second, the response is systematically tilted toward the lower end of the product ladder. When growth falls, innovation contracts most in necessity sectors; when growth rises, innovation expands most in those same sectors. Thus, inequality shocks do not simply rescale innovation uniformly—they reshape its sectoral allocation.

Our last theoretical results illustrate how variation in innovation intensity in the necessity sectors affects economic growth.

Proposition 4 *The BGE growth rate of* $N(t)$ *is determined by the innovation intensity in the most necessity sectors:*

$$g_n = \frac{1}{\theta}\frac{\beta}{1-\beta}\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right)\lim_{\tilde{i}\to 0} z(\tilde{i}).$$

The proof of Proposition 4 is given in Appendix B.7. Together with Proposition 2, which establishes that $z(\tilde{i})$ is strictly increasing in $\tilde{i}$, Proposition 4 implies that $\lim_{\tilde{i}\to 0} z(\tilde{i})$ is a lower bound for $z(\tilde{i})$ for all $0 < \tilde{i} \leq 1$, and hence the infimum of the innovation schedule. Aggregate growth is therefore pinned down by the sector with the slowest technological progress, echoing the bottleneck logic emphasized by Acemoglu et al. (2024); Jones and Tonetti (2025). In our framework, the most necessity sectors endogenously become the bottleneck. Another way to understand this result is to note that any fixed product i eventually becomes a necessity good, in the sense that $\tilde{i} \to 0$ as $t \to \infty$. Innovations in this asymptotic region apply to goods consumed by the broadest set of households, which matter most for long-run growth.

More generally, this mechanism parallels a familiar insight from the structural transformation literature (Johnston and Mellor, 1961; Gollin et al., 2007; Restuccia et al., 2008). These studies emphasize that sluggish productivity growth in agriculture delays the reallocation of resources toward higher-productivity activities and thereby depresses aggregate performance. In our setting, necessity sectors play an analogous role. When inequality dampens innovation in sectors where goods are most widely consumed, it slows the diffusion of productivity gains and the reallocation of labor toward newer, higher-value sectors, ultimately reducing the pace of variety expansion and long-run growth.

5 Quantitative Analysis

In our model, rising inequality tilts innovation away from widely consumed goods, delays product diffusion, and slows growth. In this section, we calibrate our model to quantify these forces and to evaluate three counterfactual questions: what drives product diffusion, how large the effect of inequality on growth is, and how alternative income tax schedules affect welfare.

5.1 Model Calibration

Parameter values and their identification are summarized in Table 2. The calibration separates parameters that can be pinned down directly from standard references or simple moment mappings (external calibration) from those that require solving the model and matching a small set of informative moments (internal calibration). The internal targets are chosen to discipline three objects that are central for our mechanism: aggregate growth, firms' innovation effort and size dispersion, and the speed of diffusion and entry after a product's introduction.

Externally Calibrated Parameters. We set preferences to conventional values: the discount rate is $\rho = 0.02$ and the inverse intertemporal elasticity of substitution is $\sigma = 2$, following Acemoglu et al. (2018). The elasticity of innovation with respect to R&D is set to $\alpha = 0.5$, consistent with estimates in Hall and Van Reenen (2000), Blundell et al. (2002), and Acemoglu et al. (2018). Firm exit is fixed at $\delta = 0.099$, targeting establishment exit rates in the U.S. Business Dynamics Statistics (BDS) over 2004–2016. The parameter β is disciplined by operating profitability. In the model, the average profit-to-sales ratio of intermediate producers equals $1-(1-\beta)^2$, which maps directly into the combined pre-tax profit and R&D expenditure rate reported in Akcigit and Kerr (2018). This yields $\beta = 0.056$. The hierarchy-preference parameter γ is disciplined by the long-run relation between aggregate growth and the trend in sectoral prices. The proof of Proposition 1 implies

$$g_p \;=\; -\frac{\gamma}{1-\gamma}\, g_y, \tag{25}$$

where g_y is the asymptotic growth rate of per-capita output and g_p is the asymptotic growth rate of sector-level prices $P(i,t)$. We target $g_y = 1.6\%$, the average U.S. per-capita GDP growth rate over 2010–2019, and set $g_p = -0.86\%$, the average price trend for necessity goods in our sample.[9] equation (25) then implies $\gamma = 0.343$.

Finally, we discipline the baseline income distribution using the World Inequality Database (WID) (Piketty et al., 2018). To obtain a transparent, one-parameter summary of U.S. inequality, we assume incomes follow a Pareto distribution and choose the shape parameter ϵ to match the average U.S. post-tax Gini coefficient in the 2010s, 0.484. This mapping implies $\epsilon = 1.533$.[10] Together with the external targets for U.S. aggregate growth and price trends in the 2010s, this pins down a baseline economy that matches the key moments used in our calibration and provides a coherent starting point for the counterfactual exercises.

Internally Calibrated Parameters. The remaining parameters are disciplined by matching model moments to their empirical counterparts. For transparency, each parameter is tied to a single primary target.

The parameter ϕ governs the efficiency of new final-good invention and is the main determinant of the long-run rate of variety creation. We choose $\phi = 2.987$ to match per-capita GDP growth of 1.6% in the 2010s. The parameter ψ governs incumbents' innovation efficiency. We choose $\psi = 0.223$ to match the average firm-level R&D-to-sales ratio of 4.1% reported in Akcigit and Kerr (2018). We normalize the entry-efficiency parameter to $\psi_e = 1$, which fixes the level of the mass of firms without affecting growth rates or the cross-sector slopes that are central to the paper. In the model, scaling ψ_e simply rescales the number of operating firms and the level of output, leaving balanced-growth rates and the monotone patterns across $\tilde{i}$ unchanged. This scale invariance makes ψ_e a natural normalization. The parameter ζ governs the initial productivity level in a newly introduced sector. A higher ζ makes early entry more attractive but leaves less room for subsequent expansion as the sector matures. We choose $\zeta = 0.560$ to match the average growth in

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[9] We use the 2010s for the aggregate growth target to avoid the extreme fluctuations around the 2008–2009 financial crisis. Using 2004–2016 excluding 2008–2009 yields very similar results. We define necessity goods as those whose base-year (2004) target-consumer income lies in the bottom 5% of the income distribution; in the model, the asymptotic sectoral price trend is governed by the price trend for necessities. Results are not sensitive to γ; doubling or halving γ leaves the main quantitative conclusions largely unchanged (see Appendix Section E.2).

[10] Data are from the World Inequality Database (WID): `https://wid.world/country/usa/`. We assume income is Pareto with density

$$g(l) = \begin{cases} 0, & l < \underline{l}, \\ \frac{\epsilon\, \underline{l}^{\epsilon}}{l^{\epsilon+1}}, & l \geq \underline{l}, \end{cases}$$

where $\epsilon > 1$ is the shape parameter and $\underline{l} > 0$ is the minimum income threshold. For the Pareto family, the Gini coefficient is strictly monotone in ϵ, so matching the post-tax Gini identifies ϵ uniquely. We normalize mean income to one by setting $\underline{l} = (\epsilon - 1)/\epsilon$, which implies $\mathbb{E}[l] = 1$.

Table 2: Calibration of Structural Parameters

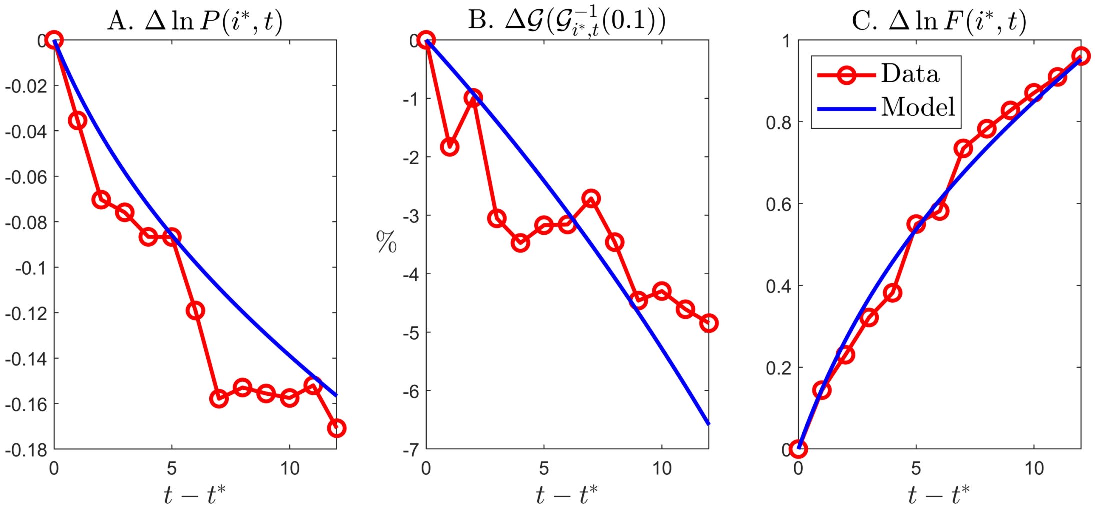
Parameter	Interpretation	Sources	Value
		External Calibrated Parameters	
ρ	Discount rate	Acemoglu et al. (2018)	0.02
σ	Intertemporal elasticity of substitution	Acemoglu et al. (2018)	2
β	Firm profitability	Akcigit and Kerr (2018)	0.056
δ	Firm exit rate	U.S. Business Dynamics Statistics	0.099
α	Elasticity of innovation to R&D	Hall and Van Reenen (2000)	0.50
γ	Degree of hierarchy preference	equation (25)	0.343
ϵ	Pareto exponent in income distribution	World Inequality Database	1.533
		Internal Calibrated Parameters	
ϕ	Productivity of inventors	GDP per-capita growth rate	2.987
ψ	Productivity of incumbents' innovation	Firm-level R&D-to-sales ratio	0.223
ψ_e	Productivity of firm entry	Normalization	1
λ	Step size of a successful innovation	Firm size dispersion	1.053
ζ	Productivity of newly-established sector	Firm number growth rate	0.560
θ	Productivity spillover on production	Product diffusion across households	0.328

Notes. "External" parameters are pinned down either by standard values in the literature or by direct mappings that do not require solving the full equilibrium (e.g., β from the implied profit-share identity). "Internal" parameters are chosen jointly so that the solved BGE matches the listed moments. Moments based on post-entry dynamics are computed as module-by-entry-year averages over the first twelve years after a module is introduced. The income distribution is parameterized as Pareto; the WID post-tax Gini in the 2010s identifies ϵ one-to-one within this family.

the number of active firms over the first twelve years after a product's introduction. The innovation step size λ governs the thickness of the right tail of the firm-size distribution. We pin down λ using the module-level top-10-percent output share, defined as the fraction of total output produced by the largest 10 percent of firms within a module. We set $\lambda = 1.053$ to match the empirical top-10-percent share of 83.40% among necessity goods in our sample, close to the value reported in Akcigit and Kerr (2018). The details of solving firm distribution dynamics are provided in Appendix E.1. The parameter θ governs the extent to which aggregate productivity improvements translate into firm-level productivity, and therefore how quickly products become affordable to lower-income households. We discipline θ using the speed at which the income position of the marginal consumer falls after entry. Empirically, we proxy the marginal consumer by bottom-decile (D1) purchasers within a module and target the average decline in their economy-wide income quantile over the first twelve years after entry. This yields $\theta = 0.328$.

Model Fit A natural test of the calibrated economy is whether it reproduces the empirical signature of diffusion: new products enter as high-end purchases and then become progressively cheaper and more widely consumed. Figure 6 compares the model with the data by tracing product-level dynamics following entry. The blue line plots the model-implied trajectory of any fixed product i^* after its invention at some time t^*, while the red line shows the corresponding trajectory in the data. All empirical series are computed as module-by-entry-year averages. For any variable X of interest, we define $\Delta X(t) \equiv X(t) - X(t^*)$ as its cumulative changes after entry. Panel A reports

Figure 6: Model fit



Notes. The figure compares the model's predictions (blue) to the data (red) for the evolution of a product module after it is introduced. Each empirical series is constructed as a module-by-entry-year average. Panel A plots the cumulative log change in the module's average price relative to the entry year. Panel B plots the change in the economy-wide income quantile of the module's bottom-decile buyers (a proxy for the marginal buyer cutoff). Panel C plots the log change in the number of active firms in the module.

log price dynamics, $\Delta \ln P(i^*, t)$. Panel B tracks the change in the overall income percentile of the 10th-percentile buyer—our proxy for the marginal buyer. Specifically, it plots $\Delta\mathcal{G}(\mathcal{G}^{-1}_{i^*,t}(0.1))$, where $\mathcal{G}_{i^*,t}$ denotes the income distribution of buyers of product i^* at time t. Panel C reports the log change in the number of active firms relative to the entry date.

The model matches the three diffusion patterns emphasized in Section 2.2. Prices fall after entry and the pace of decline decelerates with maturity: over the first twelve years, the average price decline is 17.1% in the data and 15.6% in the model, with similar early-versus-late dynamics (years 1–6: 1.98% vs. 1.65% per year; years 7–12: 0.86% vs. 0.95%). The income position of marginal buyers shifts downward in both the data and the model, indicating diffusion toward lower-income households. Firm participation rises rapidly after entry and then gradually slows; the model implies firm-number growth of 10.29% per year in years 1–6 and 5.29% in years 7–12, close to the empirical 9.70% and 6.31%. Overall, the calibrated economy captures the joint post-entry dynamics of prices, diffusion, and market structure, providing a disciplined baseline for the counterfactuals below.

5.2 Mechanism of the "Luxury-to-Necessity" Transition

Having disciplined the model to match post-entry dynamics, we use it to unpack the diffusion mechanism—why prices fall and the marginal buyer moves down the income distribution. The strategy is a clean channel decomposition within the calibrated economy. We take a single sector i and feed it alternative paths for its sectoral productivity $B(i, t)$ that selectively shut down incumbent cost-reducing innovation and entry/exit in equation (19). All other sectors remain on the BGP, so aggregate objects such as $\bar{A}(t)$, $W(t)$, and the common spillover term are held fixed.

Because sector i is measure zero, this perturbation generates no general-equilibrium feedback; any change in diffusion reflects only the sector's own productivity evolution.

Inserting $P(i,t) = \kappa_l(i,t)$ in equation (11), diffusion is pinned down by the marginal-buyer condition,

$$1 - \mathcal{G}(\ell(i,t)) = (1-\beta)^{\frac{1-2\beta}{\beta}} W(t)^{-\frac{1-\beta}{\beta}} \kappa_l(i,t)^{\frac{1-\beta}{\beta}} B(i,t), \tag{26}$$

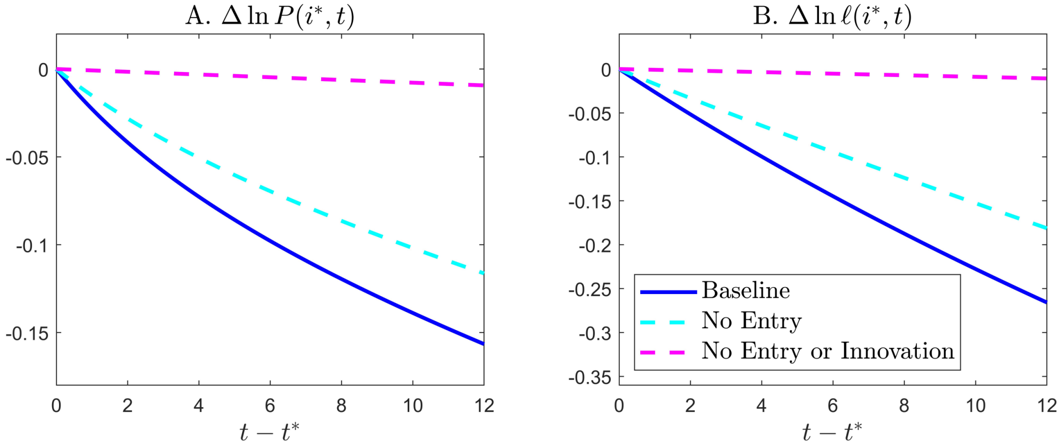
where $\ell(i,t)$ is the labor endowment of the marginal buyer, defined in equation (B.20), and $\kappa_l(i,t)$ is that buyer's reservation price. With aggregates held fixed, higher $B(i,t)$ lowers the effective price, expands coverage, and pushes the marginal buyer toward lower incomes (a lower $\ell(i,t)$).

Equation (19) implies that sectoral productivity growth can be decomposed into three terms:

$$\underbrace{S(t) = (1-\theta)\frac{1-\beta}{\beta}\frac{\dot{\bar{A}}(t)}{\bar{A}(t)}}_{\text{Common Spillovers}}, \qquad \underbrace{I_i(t) = z_i(t)\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right)}_{\text{Incumbent R\&D}}, \qquad \underbrace{E_i(t) = \frac{e(i,t)}{F(i,t)} - \delta}_{\text{Firm Entry–Exit}}.$$

The spillover term $S(t)$ is common across sectors. The term $I_i(t)$ captures incumbents' within-firm cost reductions from R&D, while $E_i(t)$ captures entry- and exit-driven changes in average sectoral efficiency. Our counterfactuals hold the common spillover component $S(t)$ fixed and shut down the other two channels sequentially.

Figure 7: Decomposition of Product Diffusion



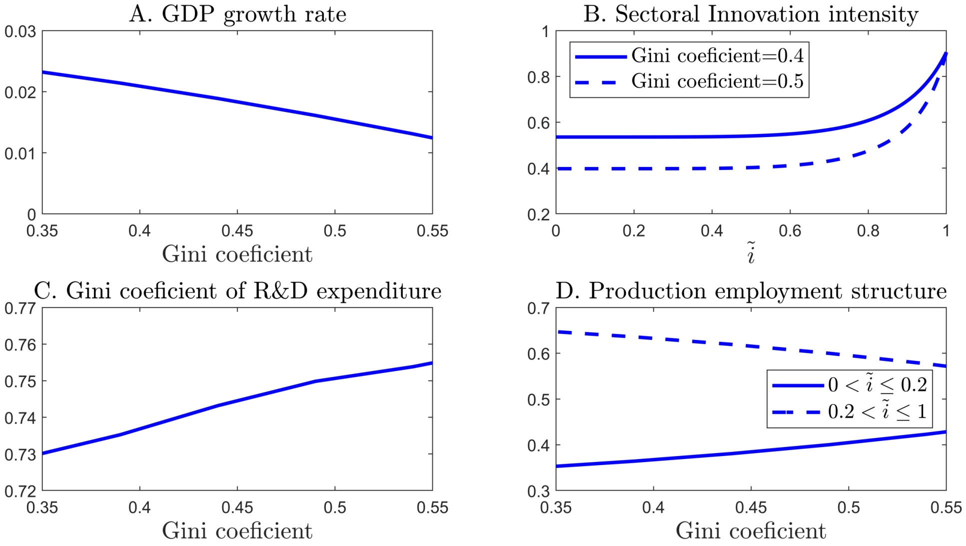
Notes. The figure traces post-entry diffusion in the calibrated economy under a nested sequence of counterfactuals that selectively remove the sector-specific forces behind productivity growth. Panel A reports the log change in product price; Panel B reports the the log change in endowment level of the marginal consumers. "Baseline" incorporates all three components in equation (19). "No Entry" sets $E_i(t) = 0$, shutting down firm dynamics. "No Entry or Innovation" additionally sets $I_i(t) = 0$, leaving only the common spillover component.

Figure 7 summarizes the implied price paths and diffusion patterns. We trace a fixed product i^* after its invention at some time t^*. Panel A shows that the steady post-entry price decline in the baseline is largely accounted for by endogenous sectoral forces: once firm dynamics are removed, the decline becomes much more muted; once both firm dynamics and incumbents' innovation are removed, prices are essentially flat. Panel B shows the corresponding movement of the marginal buyer's income (essentially $\ell(i,t)$ determined in equation (26)). In the baseline, the income of the marginal buyer falls substantially, indicating diffusion toward lower-income households. Shutting down firm dynamics markedly slows this movement, and shutting down both entry and innovation nearly eliminates it. Put differently, without incumbents' innovation and the extensive-margin adjustment of firms, a new product remains disproportionately confined to its initial, higher-income customers and fails to complete the luxury-to-necessity transition.

5.3 Income Inequality, R&D Allocation and Growth

We next study how income inequality affects economic growth through a set of comparative statics exercises in our model. Around the benchmark calibration, we vary the Pareto exponent to generate different degrees of inequality, while adjusting the minimum income threshold so that mean income remains fixed. This exercise isolates the role of redistribution in shaping demand composition and innovation incentives, holding aggregate resources constant.

Figure 8: Income Inequality, Innovation, and Long-Run Growth



Notes. The figure illustrates how changes in income inequality shift innovation incentives and long-run growth in the model. Inequality is generated by varying the Pareto exponent ϵ while adjusting the minimum income threshold to keep mean income constant. Panel A plots the implied output growth rate as inequality varies. Panel B compares the sectoral innovation schedule across the product ladder under two inequality levels. Panels C and D plot, respectively, the Gini coefficient of incumbent R&D expenditure and the production labor in necessity and non-necessity sectors.

Figure 8 summarizes the key implications. Panel A shows that higher inequality is associated with slower output growth. Quantitatively, a 0.01 increase in the Gini coefficient reduces the long-run GDP growth rate by about 0.06 percentage points. The key intuition underlying this growth effect is captured by Propositions 3 and Propositions 4. They are illustrated by Panel B, which compares the sectoral innovation schedules ($z(\tilde{i})$) under two levels of inequality (Gini coefficients of 0.4 and 0.5). As the income gap widens, moving from the solid line to the dashed line, innovation declines in all sectors, but the shift is not parallel: the contraction is largest in necessity sectors and becomes progressively more muted toward luxury sectors. This is the key reallocation force in the model. By eroding the purchasing power of the marginal low-income buyers, higher inequality compresses necessity-sector prices and profits and weakens incentives for cost-reducing innovation precisely where diffusion is the broadest, which leads to slower growth.

Panel C highlights a second, reinforcing margin: inequality raises the concentration of R&D across firms, measured by the Gini coefficient of firm-level R&D spending. This concentration is not innocuous. It reduces the efficiency of aggregate R&D and, at the same time, pulls innovative effort away from necessity sectors. When innovation slows where goods are widely consumed, resources remain tied to mature sectors, diffusion is delayed, and the economy finds it harder to sustain rapid variety expansion. Through this misallocation of innovative effort, inequality exerts an additional drag on long-run growth. This mechanism is robust to alternative labor specifications—for instance, we consider that skilled labor is used in R&D and unskilled labor in production in Appendix Section C.1. Even holding aggregate R&D labor fixed, higher inequality continues to slow growth through the concentration (misallocation) channel.

Panel D illustrates how labor becomes trapped within necessity sectors by showing the production employment structure across necessity and non-necessity sectors. We approximate necessity sectors as those with $0 < \tilde{\imath} \leq 0.2$. Specifically, Panel D plots production labor employed in necessity sectors as a share of total production labor across all sectors. As productivity growth in these sectors slows, lower production efficiency draws more labor into necessity-good production, thereby raising their employment share. This distortion hinders the spread of productivity improvements and delays the shift of labor toward more advanced sectors, thereby dampening variety expansion and long-run growth.

5.4 Welfare Effects of Redistribution

The previous subsection shows that inequality shapes both disposable income and the incentives that drive innovation and growth. Redistribution is therefore a natural policy lever in our setting. We therefore use the calibrated economy to ask how a more progressive tax-and-transfer system reshapes welfare across households.

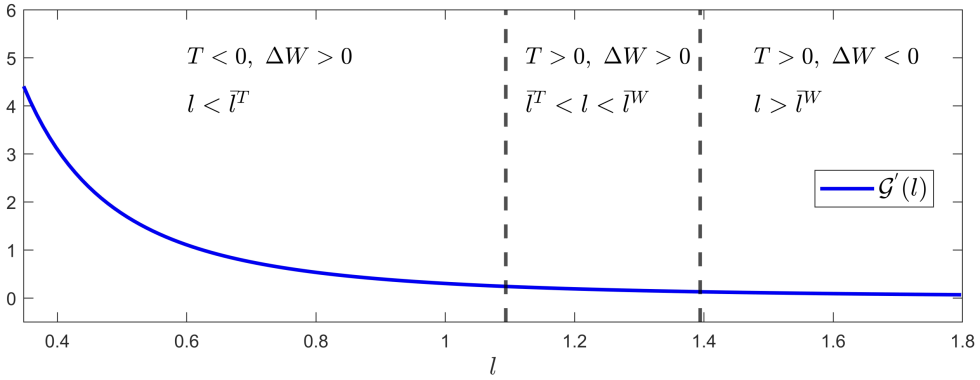
Following the public-finance literature, we represent redistribution with a parsimonious reduced-form schedule, We adopt a standard reduced-form schedule from the public-finance literature,

$$T(l) = l - \frac{l^{1-\upsilon}}{\int l^{1-\upsilon} d\mathcal{G}(l)}, \tag{27}$$

where $\upsilon \in [0,1]$ governs progressivity. The schedule is budget balanced by construction, i.e., $\int T(l) d\mathcal{G}(l) = 0$ for any υ. Since income in the model is proportional to the endowment l, $T(l)$ can be interpreted as a tax on endowments (equivalently, on pre-tax income). This functional form is widely used as a tractable representation of tax-and-transfer systems (see, e.g., Benabou (2002); Heathcote et al. (2017); Oberfield (2023)). As shown in Appendix Section E.3, if the pre-tax endowment distribution is Pareto with exponent ϵ, then the after-tax distribution remains Pareto with an adjusted exponent $\epsilon/(1-\upsilon)$, so higher υ maps into lower inequality in a transparent way.

Redistribution matters for welfare through two channels. The *level effect* reallocates contem-

Figure 9: Welfare Effects of Redistribution



Notes. The figure illustrates the distributional logic for $\upsilon = 0.8$. Households with $l < \bar{l}^T$ receive net transfers ($T(l) < 0$) and gain from both the level and growth channels. Households with $\bar{l}^T < l < \bar{l}^W$ pay positive net taxes but still gain because the growth effect dominates. For households with $l > \bar{l}^W$, the level loss from taxation outweighs the growth gain, so $\Delta W < 0$. The cutoff $\bar{l}^T$ solves $T(\bar{l}^T) = 0$, and $\bar{l}^W$ solves $\Delta W(\bar{l}^W) = 0$.

poraneous consumption: transfers relax the budgets of low-income households and tighten those of high-income households. The *growth effect* operates through the balanced-growth rate g_c: by compressing inequality, a more progressive schedule shifts innovation toward necessity sectors, speeds diffusion, and raises long-run growth. The growth effect benefits all households, while the level effect changes sign across the distribution. Figure 9 illustrates the logic by partitioning households into three regions. For low-income households with $l < \bar{l}^T$, taxes are negative ($T(l) < 0$), so they are net recipients: both the level effect and the growth effect raise welfare, implying the largest gains in the lower tail. For middle-income households with $\bar{l}^T < l < \bar{l}^W$, taxes are positive ($T(l) > 0$) but the burden is modest under a progressive schedule; here the positive growth effect dominates, so welfare still rises on net. For high-income households with $l > \bar{l}^W$, the tax burden is largest; the negative level effect outweighs the growth effect, and welfare falls.

We now quantify these welfare consequences across alternative degrees of progressivity. Table 3 reports outcomes for three values of υ. Welfare is measured using a *compensating endowment variation*: the percentage increase in endowment in the pre-policy equilibrium that makes a household indifferent between the pre-policy allocation and the allocation under the progressive schedule.

The first column reports a benchmark policy that raises progressivity to $\upsilon = 0.16$, which reduces the post-tax Gini from 0.484 to 0.376, roughly returning inequality to its 1970s level. Under this policy, almost the entire population ($l < \bar{l}^W$, about 99.5%) experiences sizable welfare gains, while welfare losses are concentrated among a very small upper tail ($l > \bar{l}^W$, about 0.5%). The implied tax rates highlight why: transfers are large for the bottom and middle of the distribution, yet positive net taxes are concentrated among top earners. In other words, a relatively small tax burden on a thin upper tail finances broad-based gains for the rest of the population, combining direct redistribution with faster long-run growth. The remaining columns consider lower progressivity ($\upsilon = 0.10$ and $\upsilon = 0.05$). As expected, both redistribution and the growth response weaken, but the qualitative pattern remains: welfare gains accrue to nearly all households, with losses confined to

Table 3: Welfare Effects of Redistribution under Alternative Tax Progressivity

	Progressivity parameter υ		
Outcome	0.16	0.10	0.05
Inequality (post-tax)			
Pre-policy Gini	0.484	0.484	0.484
Post-policy Gini	0.376	0.416	0.449
Net tax rates (percent of income)			
5th percentile household	−29.62	−18.34	−9.18
Median (50th percentile)	−21.13	−13.49	−6.92
95th percentile household	5.01	2.34	0.82
Growth effects (per-capita GDP growth rate, %)			
Pre-policy growth	1.64	1.64	1.64
Post-policy growth	2.21	2.02	1.84
Welfare effects (compensating endowment change, %)			
Share with gains ($l < \bar{l}^W$)	99.40	99.46	99.51
Avg. welfare change for gainers	56.89	35.77	18.06
Share with losses ($l > \bar{l}^W$)	0.60	0.54	0.49
Avg. welfare change for losers	-14.82	-10.597	-6.00

Notes. The table reports outcomes in the calibrated economy under the reduced-form tax-and-transfer schedule in equation (27). A higher υ implies a more progressive schedule and therefore a larger reduction in post-tax inequality. "Net tax rate" is $T(l)/l$ evaluated at the indicated income percentiles; negative values indicate net transfers. Welfare changes are measured as compensating endowment changes: the percentage increase in pre-policy endowment that makes a household indifferent between the pre-policy equilibrium and the post-policy equilibrium. The cutoff $\bar{l}^W$ denotes the income threshold at which the welfare change is zero; households with $l < \bar{l}^W$ gain and those with $l > \bar{l}^W$ lose.

the very top. These results show that modest increases in progressivity can deliver sizable welfare gains in the calibrated economy, for the same reason inequality matters here: they strengthen demand at the necessity margin and thereby redirect innovation toward the sectors where diffusion is broadest and growth returns are highest.

6 Conclusion

This paper links inequality to growth through a simple but underappreciated margin: the speed at which innovations diffuse from luxuries to necessities. Using micro evidence that connects households' purchases to firms' innovative activity and market outcomes, we document a systematic life-cycle pattern within product markets. As products mature, prices fall, adoption shifts toward lower-income consumers, and market structure becomes less concentrated. Yet innovation is disproportionately oriented toward high-income demand, and greater exposure to rising inequality is associated with weaker innovation among firms serving the mass market.

To interpret these facts, we develop a tractable endogenous-growth model with hierarchical demand, incumbent productivity-improvement R&D, and endogenous entry. These ingredients generate a tight link between the income distribution and the composition of innovative effort. Our model shows that inequality is not only a distributional outcome, but also reallocates market size across segments and the private returns to innovation. The policy implication is therefore sharper than “redistribute to raise welfare.” By sustaining the mass market for widely consumed goods, progressive tax-and-transfer policies can tilt innovation incentives back toward cost reductions that diffuse broadly, speeding the luxury-to-necessity transition. In this sense, reducing post-tax inequality can deliver a double dividend: it improves equity today while fostering innovation that raises broad-based living standards over time.

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Online Appendix

Contents of the Appendix

Appendix A provides additional data details and empirical facts.

Appendix B presents proofs in the baseline model.

Appendix C checks the robustness of the theoretical results.

Appendix D describes the numerical algorithm used in computation.

Appendix E reports robustness checks for the quantitative analysis.

A Appendix: Data and Empirical Facts

A.1 NielsenIQ Homescan Consumer Panel

To ensure data quality, Nielsen implements multiple validation checks, including a quarterly coverage monitor and weekly reviews of sample representation. Nielsen further organizes product modules—roughly 1,000 in total—into a hierarchical structure of approximately 100 product groups and 10 product departments. Figure A.1 illustrates this hierarchy. At the broadest level, products are assigned to departments (e.g., *Health & Beauty Care*), which contain narrower product groups (e.g., *Hair Care*). Each group is further divided into product modules that capture relatively close substitutes (e.g., *Shampoo*). At the most granular level, products are distinguished by their UPCs, which uniquely identify individual items such as a specific brand and package size of baby bottle.

Figure A.2 illustrates a representative "Shampoo" market. Within this product module, several well-known brands, including Pantene, Head & Shoulders, and Paul Mitchell, offer products that are close substitutes for consumers. Despite the nuances in package size (i.e., each bottle of shampoo has slightly different ounces), these shampoos exhibit a high degree of similarity in their design and utility. In other words, the high comparability of UPCs within the same market provides a unique empirical setting for the subsequent analyses at the product module level.

Figure A.3 highlights the spatial and distributional patterns of consumer expenditures. Higher spending is concentrated in coastal states, while lower levels appear more frequently in inland regions of the Midwest. Expenditures by poor consumers are markedly lower than those of rich consumers, yet both groups show steady growth over time. In 2016, for example, state-level expenditures ranged from roughly \$50 million to over \$6 billion among poor consumers, and from about \$400 million to more than \$22 billion among rich consumers. The substantial variation across consumer groups, regions, and time creates a rich setting for analyzing how local consumption patterns shape economic outcomes.

A.2 BvD Orbis Database

To track the detailed production activities and performance of our sample GS1 firms, we incorporate business information from the Bureau van Dijk (BvD) Orbis database. Orbis provides comprehensive financial data on both listed and unlisted firms worldwide. We use a string-matching algorithm, supplemented by manual validation, to match GS1 company names with Orbis entries, achieving an 86.38% success rate. Such high matching rate further validates the representativeness of Orbis database and aligns with recent research findings such as Bajgar et al. (2020).

In Orbis, the gross outputs is measured as the operating revenues. Capital is proxied by the tangible fixed assets. Labor is measured as the number of employees. Additionally, Orbis provides the cost of employees for each firm. Intermediate inputs are calculated as the difference between turnover and added value (Bajgar et al., 2020). Although the material cost that we can directly obtain from Orbis Database represents a notable part of intermediate inputs, its values for most of the sample firms are missing. Finally, we can directly obtain investment information about our sample firms.

Table A.1 reports the distribution of BvD Orbis matched firm count across NAICS four-digit industries (2022 classification). For brevity, we report only industries with more than 100 firms. Panel (a) of Appendix Figure A.4 maps the distribution of BvD Orbis matched sample firm count across U.S. states, showing concentrations in major economic centers such as California, New York, Florida, and Texas. Taken together, the breadth of industry and geographic coverage indicates that our sample firms are broadly representative of the U.S. economy.

A.3 Patent–Firm Matching in GS1 Data

Matching Procedure A crucial step in linking innovation outcomes to firms in the GS1 data is to match patent assignee names with firm names. Using Python's *FuzzyWuzzy* package, we implement an approximate matching algorithm based on the Levenshtein edit distance. This algorithm measures the similarity between two strings by calculating the minimum number of single-character edits-insertions, deletions, or substitutions—required to transform one string into the other (Levenshtein, 1966). Compared with exact string matching, this approach successfully links many pairs of slightly different names that refer to the same entity.[11] To reduce the risk of false positives, we apply a high similarity threshold of 0.9 and cross-validate matches using alternative approaches, including vector decomposition and phonetic algorithms.[12] Finally, the automated procedure is supplemented with manual verification to ensure the accuracy of the final matches.

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[11] For instance, the algorithm can reconcile "Honeywell International," "Honeywelll International Inc.," and "Honeywell International, Inc." with the GS1 entry "Honeywell International Inc."

[12] No single matching algorithm is universally superior; each has strengths and limitations. To mitigate sensitivity to the choice of method, we further combine several widely used techniques from the matching literature (Raffo and Lhuillery, 2009). Specifically, we employ *scotoken* and *token_soundex* within Stata's *matchit* package to filter implausible pairs, and we complement the edit-distance approach with the 2-gram algorithm available in Stata's *reclink2* package, which performs well with large string permutations in alphabetic languages (Thoma et al., 2010).

Validation The matching procedure delivers a close correspondence between our sample and the USPTO universe. First, the geographic and firm-level patterns reinforce the validity of the match. Panel (b) of Appendix Figure A.4 shows that patents in the sample are concentrated in the same innovation hubs highlighted in the literature—California and Massachusetts in particular—where clusters of firms drive technological progress. Panels (a) and (b) of Appendix Figure A.5 confirm this interpretation: both the sample and the USPTO universe display strikingly similar, right-skewed distributions, indicating that the overall shape of patenting activity is preserved. Similarly, Panel (c) of Appendix Figure A.5 summarizes firm-level patent counts, showing that the lower tail of the distribution (Q1) aligns almost exactly with USPTO assignees, while the median and upper quartile (Q3) are slightly higher. This upward shift is consistent with our consolidation of assignee names with minor spelling differences into single firms, which raises counts without altering the underlying distribution. Finally, Panel (d) of Appendix Figure A.5 lists the most patent-intensive firms, led by global technology leaders such as LG Electronics, Microsoft, General Electric, and Apple. The presence of these multinationals at the top of the distribution underscores that the matching procedure not only reproduces the broad structure of the USPTO data, but also captures the firms most central to U.S. innovation.

A.4 Robustness Check

We use the price level (rather than ln price) for Figure 2 because some observations are recorded as zero in the Nielsen data. These zeros typically reflect promotional transactions such as buy-one-get-one-free offers or free samples, where the item carries no incremental cost. As robustness, we re-estimate the specification using the inverse hyperbolic sine transformation, which accommodates zeros in semi-positive variables (Doran et al., 2022). This approach deliver qualitatively similar results (Appendix Figure A.8).

The diffusion and price dynamics are robust to a range of alternative samples and measurements. We (i) restrict the sample to firms that patent at least once during the period; (ii) move from modules to UPCs, dropping UPCs with version changes and adding UPC fixed effects; (iii) use size-adjusted module prices to address within-module package-size variation; and (iv) document direct evidence of technological upgrading using firm-level TFP estimates (computed with *prodest*). See Appendix Figures A.6–A.9 and Appendix A.2. Standard errors are clustered at the firm level throughout.

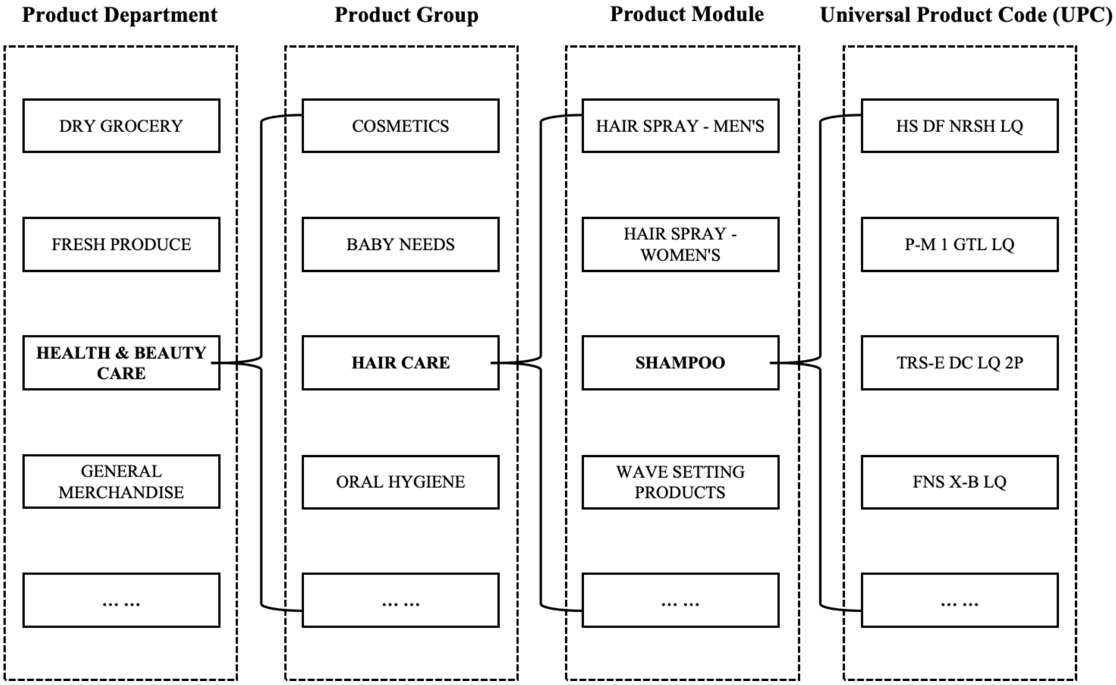
For Figure 3, high-income consumers are defined in the Nielsen data description. The pattern is robust to alternative cutoffs (40th and 60th percentiles); see Table A.2. Patent counts mask heterogeneity in patent value. As a quality adjustment, we also use claim-weighted patent counts following Aghion et al. (2019); the rich–poor gap remains essentially unchanged (Appendix Table A.3). Appendix Figures A.10–A.13 provide a continuous analogue using binned scatters, showing a robust positive relationship between innovation intensity and the expenditure share of high-income consumers within each year.

Figure A.14 reports a similar pattern to Figure 4 using concentration ratios: $CR4$ and $CR20$, defined as the combined revenue share of the top 4 and top 20 firms in a module-year, respectively.

Figure A.15 shows that our adopted Gini series and the Theil and GE indices are highly correlated. Figure A.16 shows that our Gini series closely tracks the U.S. Census Bureau counterpart.

A.5 Complementary Figure

Figure A.1: Nielsen Product Hierarchy from Departments to UPCs



Notes. This figure illustrates the product hierarchy defined by Nielsen, from broad departments to specific Universal Product Codes (UPCs).

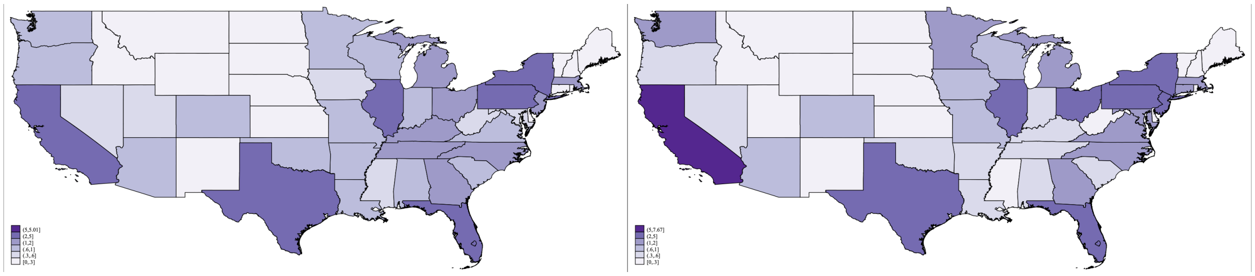
Figure A.2: UPCs of Different Brands within Shampoo Market

(a) Pantene Pro-V, 12.6 OZ (b) Matrix, 13.5 OZ (c) Head & Shoulders, 13.5 OZ

(d) Paul Mitchell, 10.14 OZ (e) Garnier Fructis, 13.0 OZ (f) TIGI, 6.76 OZ

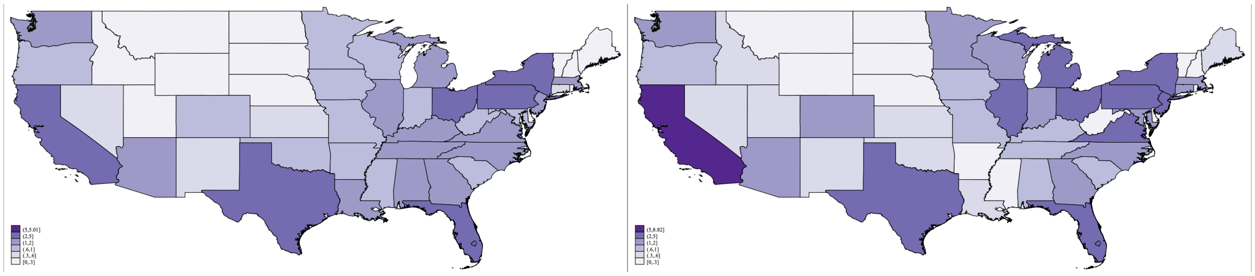
Notes. This figure illustrates several representative Universal Product Codes (UPCs) within the product module Shampoo. Pictures of UPCs can be downloaded from `https://www.upcitemdb.com/`. Despite nuances in package size, UPCs of different product brands show large similarity in product design and utility.

Figure A.3: Geographic Distribution of Consumer Expenditures by Income Group, 2004–2016 (Billions of USD)



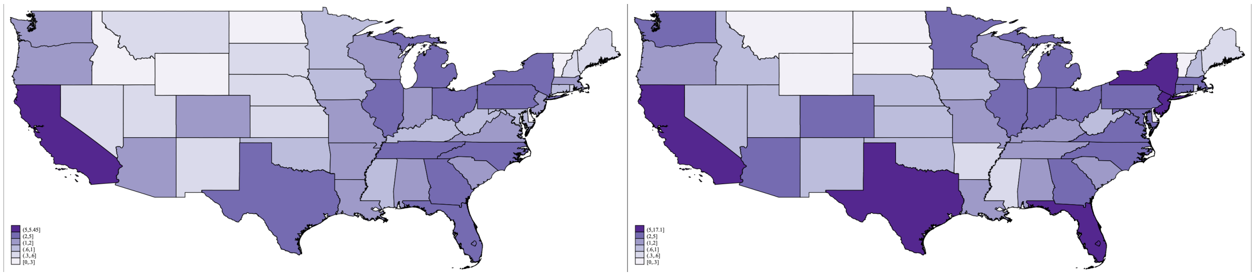
(a) Poor Consumers, 2004

(b) Rich Consumers, 2004



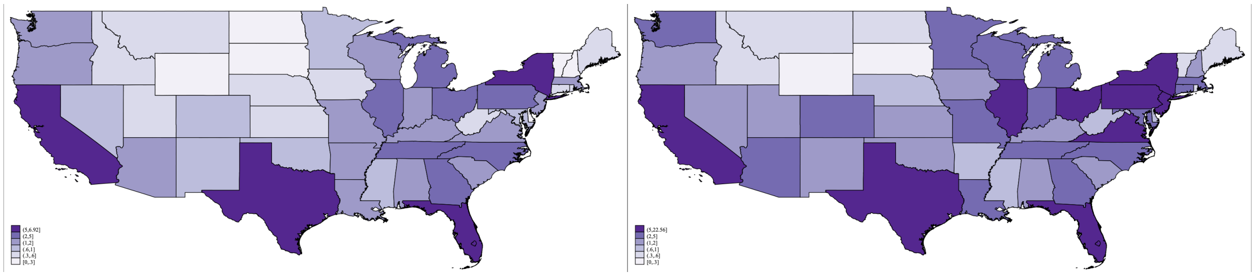
(c) Poor Consumers, 2008

(d) Rich Consumers, 2008



(e) Poor Consumers, 2012

(f) Rich Consumers, 2012

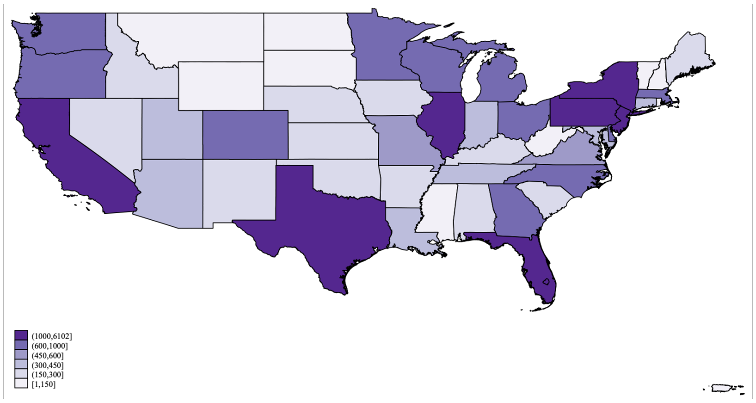


(g) Poor Consumers, 2016

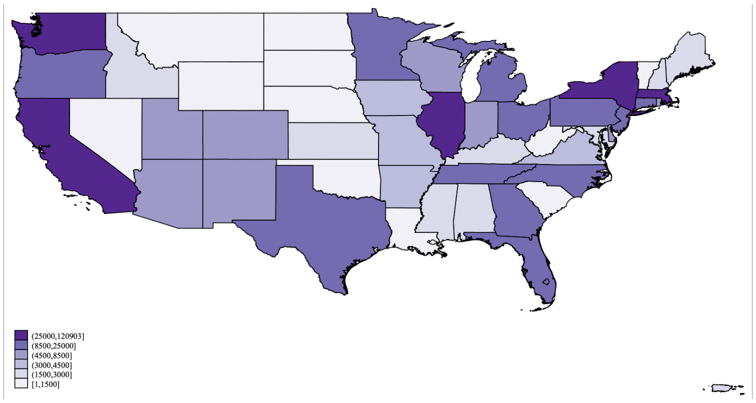
(h) Rich Consumers, 2016

Notes. The figure maps projection factor–weighted consumer expenditures (in billions of U.S. dollars) across states for 2004, 2008, 2012, and 2016. Panels (a), (c), (e), and (g) display expenditures of poor consumers (household income below $20,000, roughly the bottom decile). Panels (b), (d), (f), and (h) display expenditures of rich consumers (household income in the top income bracket, roughly the top decile). Expenditures are consistently higher among rich consumers and more concentrated in coastal states, while poorer consumers account for lower levels of spending, especially in inland regions.

Figure A.4: Geographic Distribution of BvD–Orbis Firms and Innovation in the United States, 2004-2016



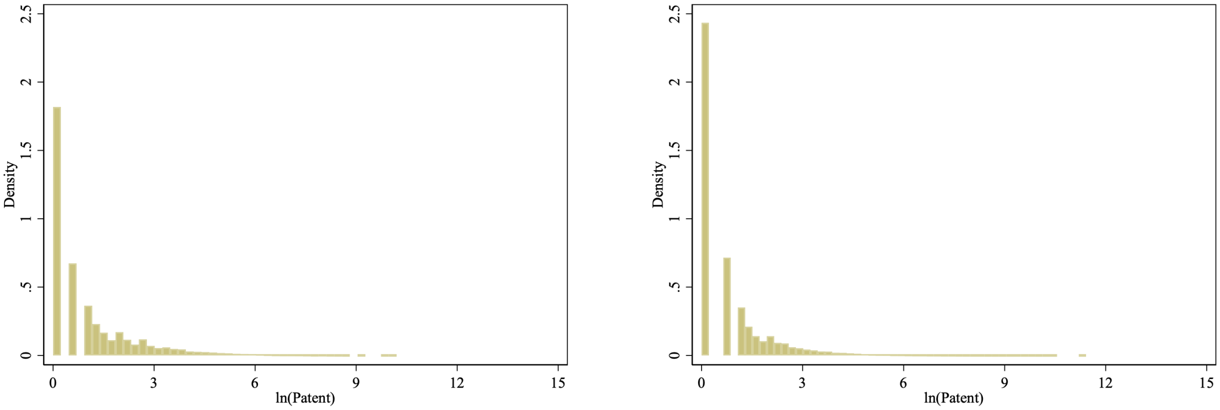
(a) BvD Orbis Matched Sample Firm Count



(b) Total Patent Count Filed by BvD Orbis Matched Sample Firms

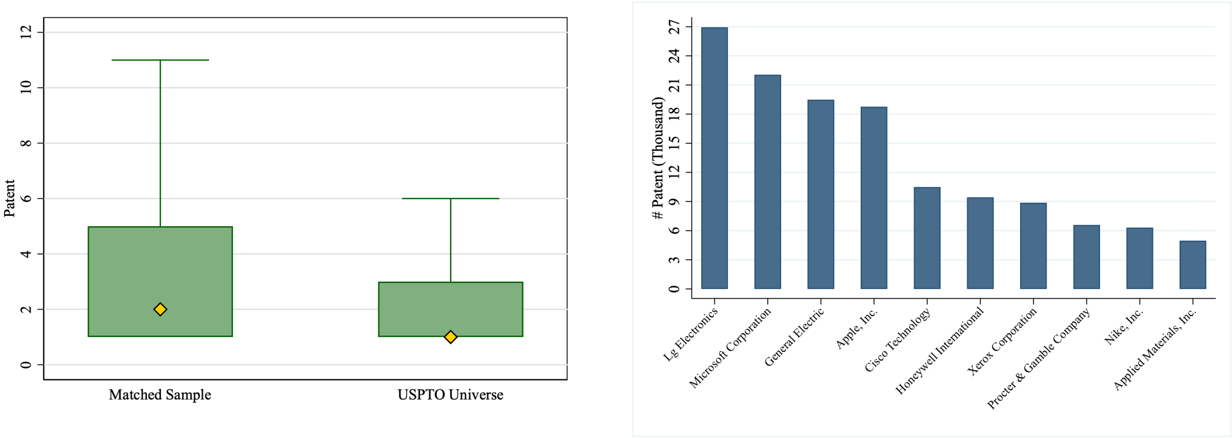
Notes. Panel (a) shows the spatial distribution of BvD Orbis matched firm count in our sample based on their headquarters' location as reported in the BvD Orbis database. The majority of firms are concentrated in economically active states such as California, New York, Florida, and Texas, while relatively few are located in sparsely populated states. For clarity of presentation, Hawaii, Alaska, and Guam are omitted; together they account for only 0.51% of the sample. Panel (b) maps the distribution of patents filed by firms in our Bvd Orbis matched sample across U.S. states, based on the location of firm headquarters. Patent activity is concentrated in economically dynamic states such as California, New York, Massachusetts, and Texas. For clarity of presentation, Hawaii, Alaska, and Guam are omitted, as together they account for only 0.33% of total patents.

Figure A.5: Validation of Patent Match, 2004-2016



(a) Firm-Level Patent Count Distributions: Matched Sample

(b) Firm-Level Patent Count Distributions: USPTO Universe

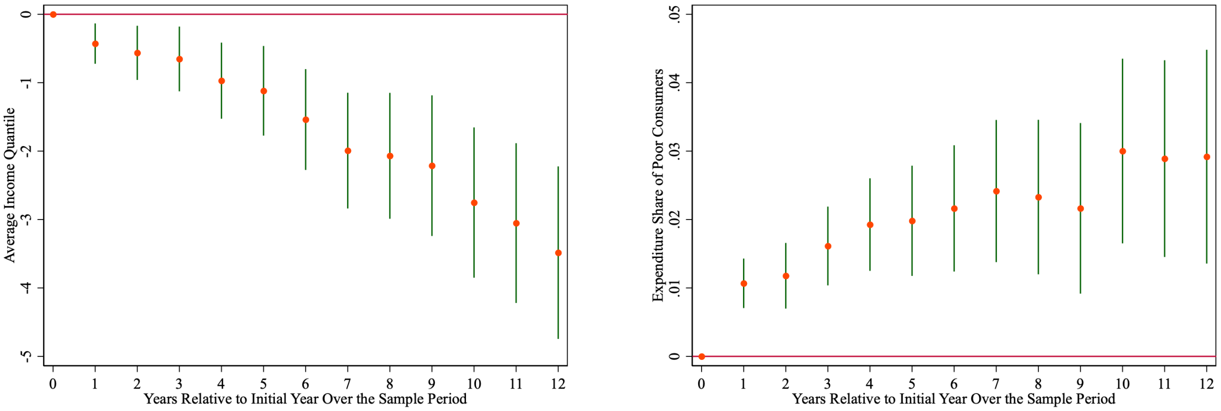


(c) Box Plots of Firm-Level Patent Counts: Matched Sample vs. USPTO Universe

(d) Top Ten Patent-Holding Firms in the Matched Sample

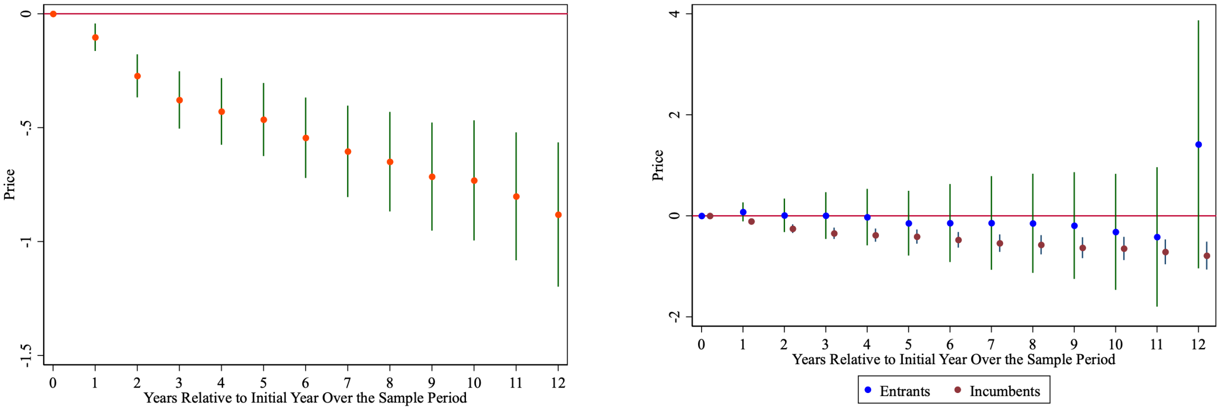
Notes. The figure offers a comprehensive validation of the quality of our patent matching. Panels (a) and (b) compare the distribution of firm-level patent counts (log scale) in the matched sample of firms and the full USPTO universe of assignees. Specifically, Panel (a) reports the distribution for the matched sample, while Panel (b) shows the universe of USPTO assignees. Both distributions are right-skewed, with broadly similar shapes, indicating that the matched sample captures the overall structure of patenting activity. Panel (c) presents box plots of firm-level patent counts for our matched sample ("Matched Sample") and the full universe of USPTO assignees ("USPTO Universe"). Diamonds indicate medians. For clarity, extreme values are excluded from the plots. Panel (d) lists the ten firms in our matched sample with the highest number of granted patents over the sample period. Patent counts are aggregated at the firm level after consolidating assignee names. High-technology multinationals such as LG Electronics, Microsoft, General Electric, and Apple dominate the ranking, underscoring the concentration of patenting activity among global leaders.

Figure A.6: Robustness: Product Diffusion Across the Income Distribution, 2004-2016



(a) Decline in Buyers' Income

(b) Growing Share of Low-Income Buyers

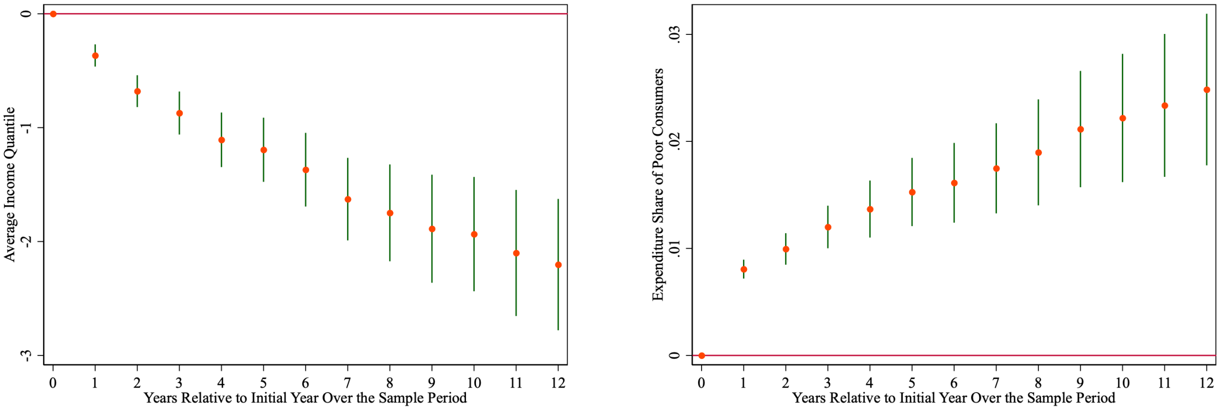


(c) Falling Prices Over Product Age

(d) Falling Prices: Entrants vs. Incumbents

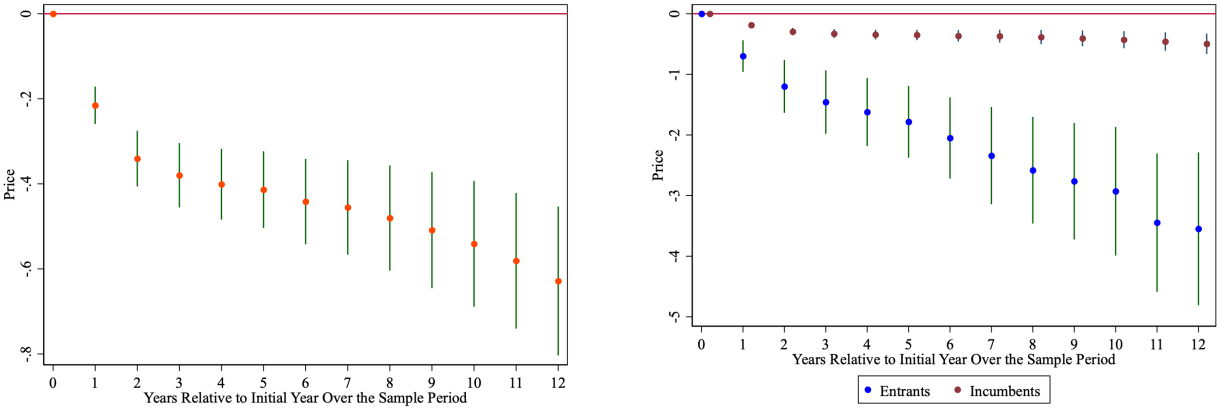
Notes. We further restrict the analysis to innovative firms by excluding those that never filed a patent during the sample period and re-estimate equation (1). Each point plots the estimated coefficient from a regression of product-level outcomes on the cumulative years since a product module had sales records during the sample period. The x-axis measures years relative to initial year, and the y-axis shows the estimated change in the corresponding outcome relative to initial year. All regressions include firm–product, firm–year, and product–year fixed effects, with standard errors clustered at the firm level. In Panel (a), the dependent variable is the average income quantile of a product's buyers, showing whether its customer base shifts toward lower-income consumers as it matures. In Panel (b), the dependent variable is the expenditure share of the bottom income tercile, reflecting the growing importance of poorer consumers in total spending. Panel (c) uses the annual price to track how average prices evolve after market entry. Panel (d) replicates falling price pattern using entrants and incumbents subsamples, respectively. Solid dots represent point estimates, and vertical bars denote 95% confidence intervals based on robust standard errors.

Figure A.7: UPC-level Analyses: Product Diffusion Across the Income Distribution, 2004-2016



(a) Decline in Buyers' Income

(b) Growing Share of Low-Income Buyers

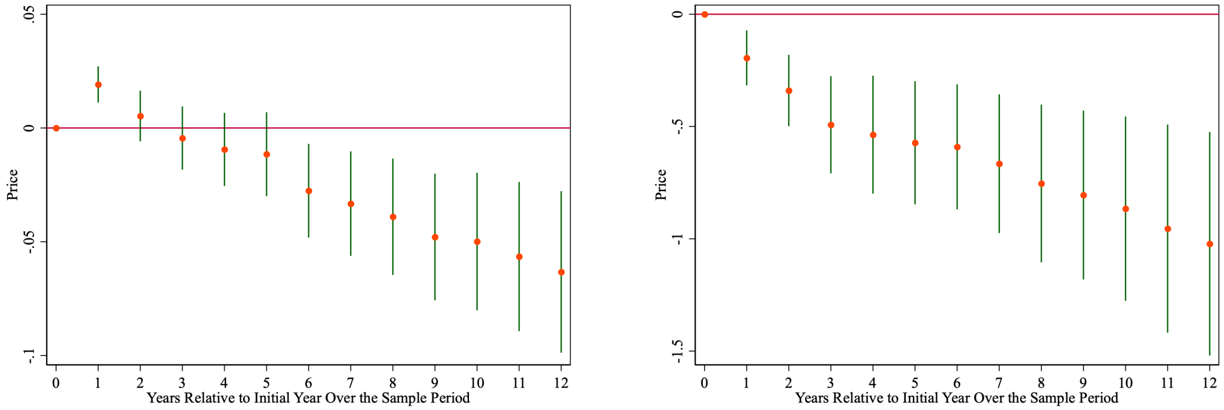


(c) Falling Prices Over Product Age

(d) Falling Prices: Entrants vs. Incumbents

Notes. We replicate empirical analyses of Fact 1 at the most disaggregated level (i.e., firm-UPC-year level). Each point plots the estimated coefficient from a regression of product-level outcomes on the cumulative years since a product module had sales records over the sample period. The x-axis measures years relative to initial year during the sample period, and the y-axis shows the estimated change in the corresponding outcome relative to entry year. All regressions include firm–product, firm–year, and product–year fixed effects, with standard errors clustered at the firm level. In addition, we control UPC fixed effects. In Panel (a), the dependent variable is the average income quantile of a product's buyers, showing whether its customer base shifts toward lower-income consumers as it matures. In Panel (b), the dependent variable is the expenditure share of the bottom income tercile, reflecting the growing importance of poorer consumers in total spending. Panel (c) uses the annual price to track how average prices evolve after market entry. Panel (d) replicates falling price pattern using entrants and incumbents subsamples, respectively. Solid dots represent point estimates, and vertical bars denote 95% confidence intervals based on robust standard errors.

Figure A.8: Additional Evidence: Falling Prices Pattern, 2004-2016

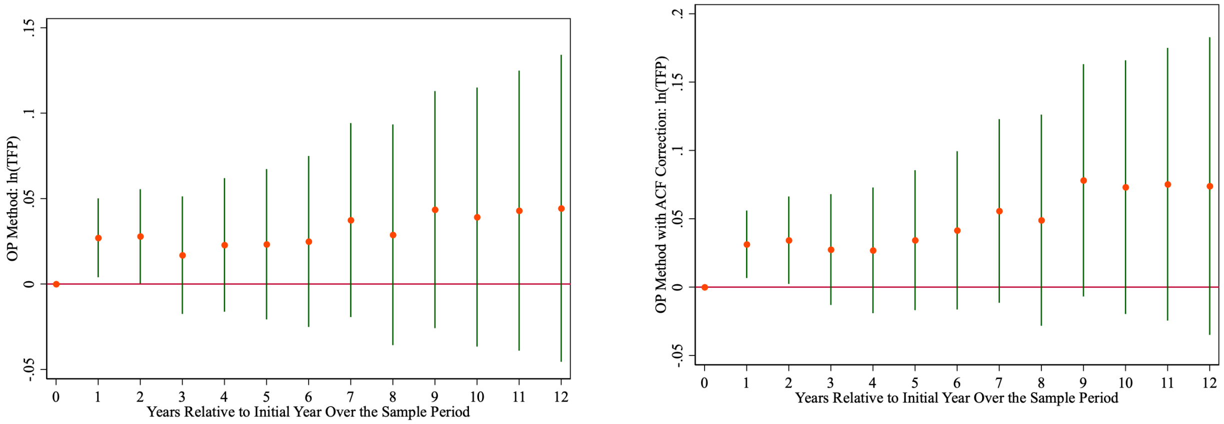


(a) Falling Prices (in Inverse Hyperbolic Sine Form) Over Product Age

(b) Falling Module Size-adjusted Prices Over Product Age

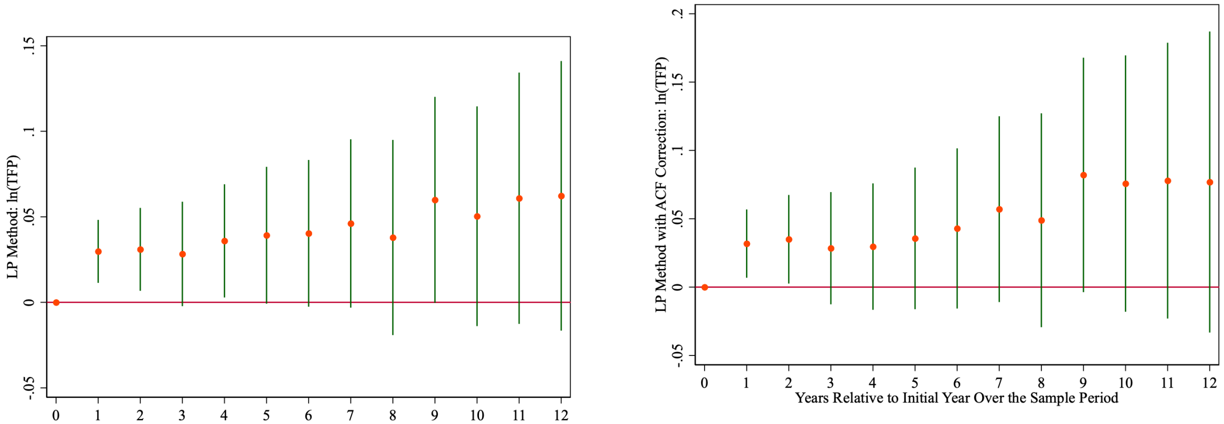
Notes. This battery of robustness checks (1) use alternative transformation of product price and (2) address concerns regarding nuances in product size of different UPCs within the same product module. The x-axis measures years relative to initial year of market sales over the sample period, and the y-axis shows the estimated change in the corresponding outcome relative to initial year. Both regressions include firm–module, firm–year, and module–year fixed effects, with standard errors clustered at the firm level. Panel (a) uses inverse hypoebolic sine transformation of level price variable. This result shows a similar picture of falling prices over product age as the baseline results. Panel (b) uses the annual product module size-adjusted price to track how average prices evolve after market entry. Solid dots represent point estimates, and vertical bars denote 95% confidence intervals based on robust standard errors. All these results remain qualitatively unchanged compared to our baseline estimates.

Figure A.9: Further Evidence on Firm Innovation: Growing TFP Pattern, 2004-2016



(a) Olley and Pakes (OP) Methodology

(b) Olley and Pakes (OP) Methodology with Ackerberg, Caves and Frazer (ACF) Correction

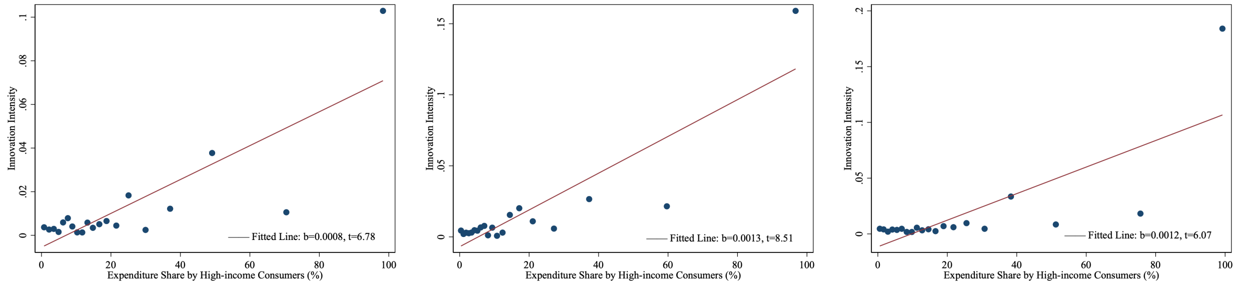


(c) Levinsohn and Petrin (LP) Methodology

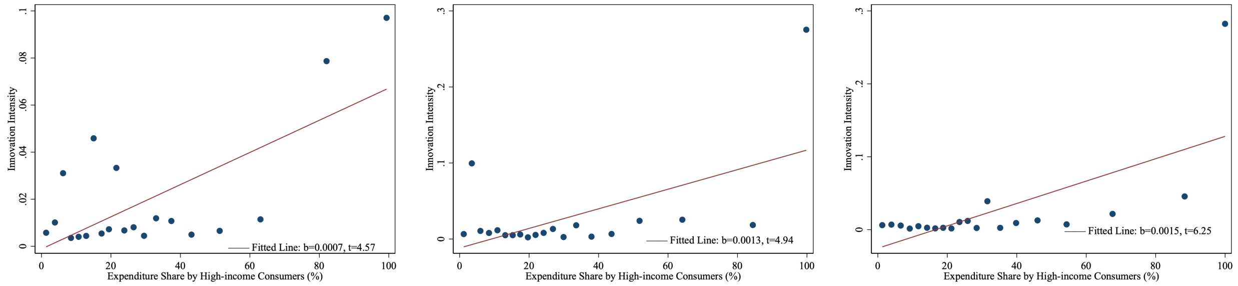
(d) Levinsohn and Petrin (LP) Methodology with Ackerberg, Caves and Frazer (ACF) Correction

Notes. We provide additional evidence on firm innovation by documenting growing firm TFP pattern. Each point plots the estimated coefficient from a regression of firm-level TFP on the years relative to initial year of market sales over the sample period. The x-axis measures years relative to initial year of market entry over the sample period, and the y-axis shows the estimated change in the corresponding outcome relative to initial year. All regressions include firm and year fixed effects, with standard errors clustered at the firm level. In Panel (a), the dependent variable is the firm TFP calculated using Olley and Pakes (OP) Methodology. In Panel (b), the dependent variable is the firm TDP calculated using Olley and Pakes (OP) Methodology with Ackerberg, Caves and Frazer (ACF) Correction. In Panel (c), the dependent variable is the firm TDP calculated using Levinsohn and Petrin (LP) Methodology. In Panel (d), the dependent variable is the firm TDP calculated using Levinsohn and Petrin (LP) Methodology with Ackerberg, Caves and Frazer (ACF) Correction. Solid dots represent point estimates, and vertical bars denote 95% confidence intervals based on robust standard errors.

Figure A.10: Patent per Expenditure of Thousand U.S. Dollars and Expenditure Share of High-income Buyers, 2004-2009



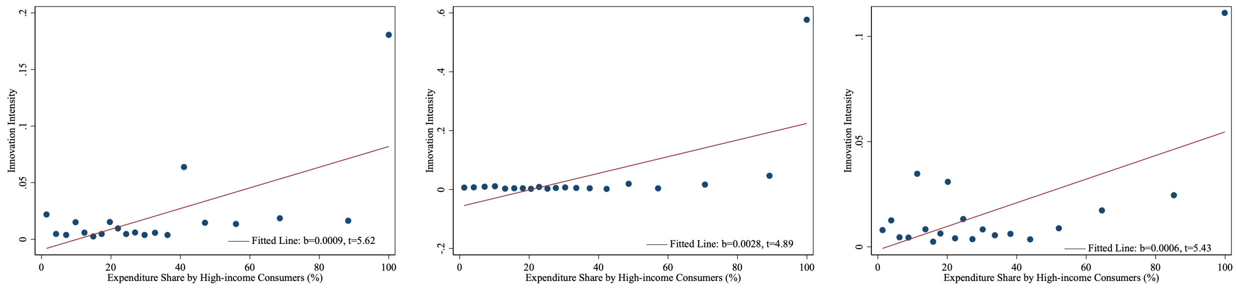
(a) Year 2004 (b) Year 2005 (c) Year 2006



(d) Year 2007 (e) Year 2008 (f) Year 2009

Notes. We provide additional evidence on heterogeneous firm innovation intensity pattern from 2004 to 2009 by plotting the binned scatters between expenditure share of high-income consumers and innovation intensity proxied by patent count per expenditure of thousand U.S. dollars. We exclude those firm observations which have no sales records from high-income buyers. The number of bins is 20 as default. Each point represents a sample firm. The slope of the fitted line, along with its high statistical significance level proxied by t-statistics, indicates that firms targeting the rich buyers engage in more innovation activities.

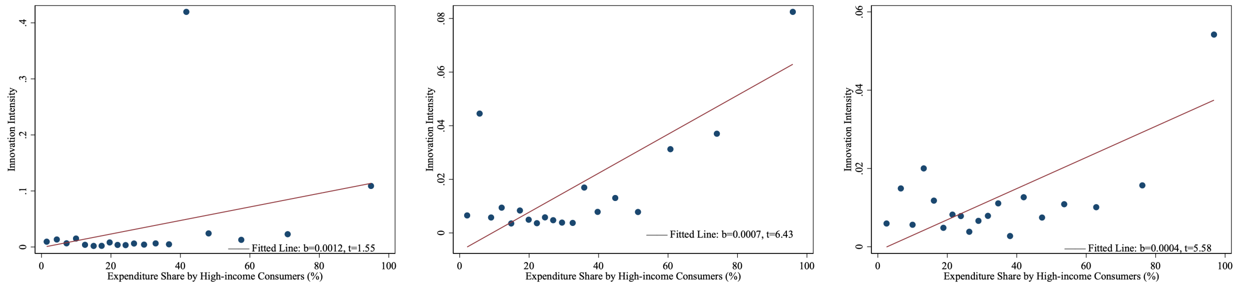
Figure A.11: Patent per Expenditure of Thousand U.S. Dollars and Expenditure Share of High-income Buyers, 2010-2016



(a) Year 2010

(b) Year 2011

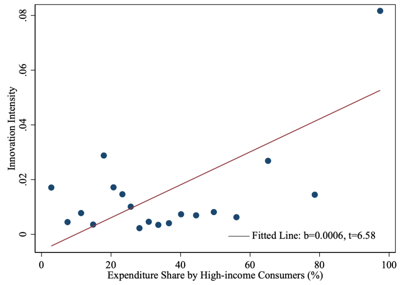
(c) Year 2012



(d) Year 2013

(e) Year 2014

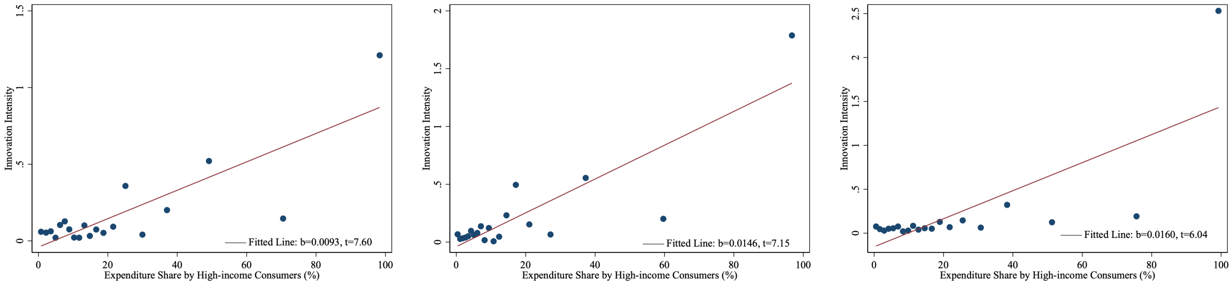
(f) Year 2015



(g) Year 2016

Notes. We provide additional evidence on heterogeneous firm innovation intensity pattern from 2010 to 2016 by plotting the binned scatters between expenditure share of high-income consumers and innovation intensity proxied by patent count per expenditure of thousand U.S. dollars. We exclude those firm observations which have no sales records from high-income buyers. The number of bins is 20 as default. Each point represents a sample firm. The slope of the fitted line, along with its high statistical significance level proxied by t-statistics, indicates that firms targeting the rich buyers engage in more innovation activities.

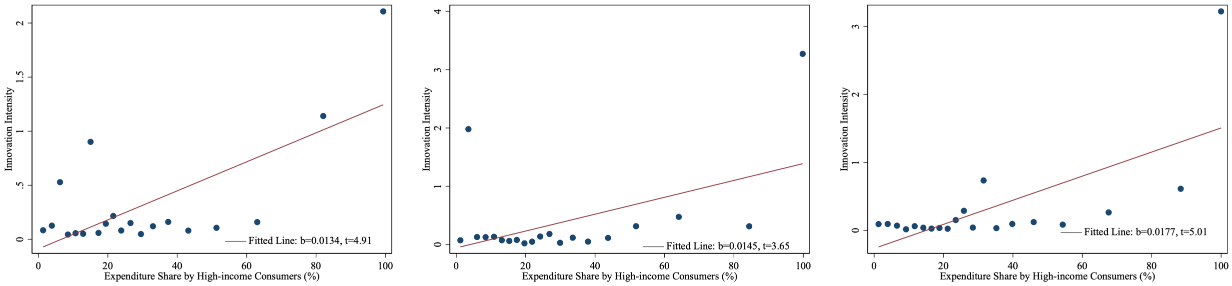
Figure A.12: Claims-weighted Patent per Expenditure of Thousand U.S. Dollars and Expenditure Share of High-income Buyers, 2004-2009



(a) Year 2004

(b) Year 2005

(c) Year 2006



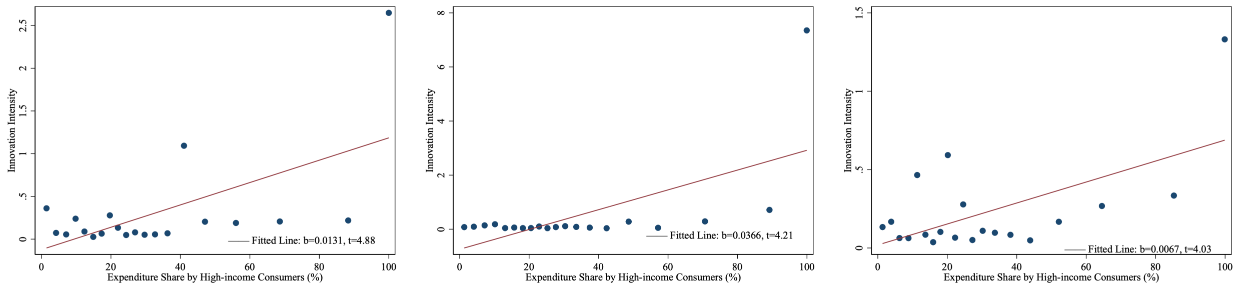
(d) Year 2007

(e) Year 2008

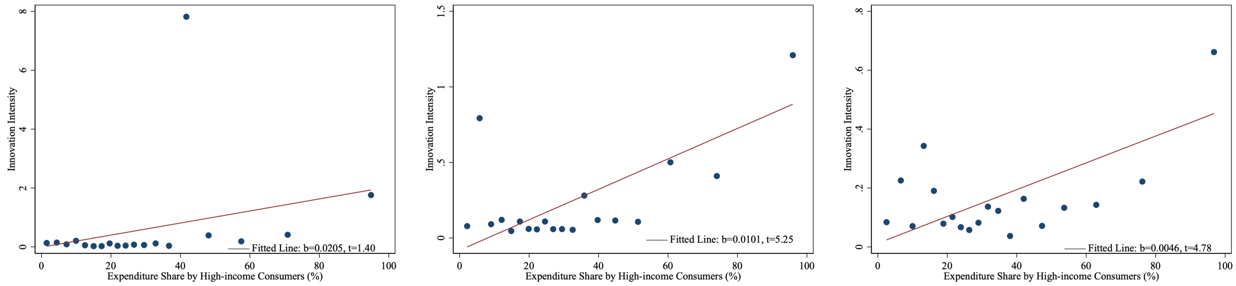
(f) Year 2009

Notes. We provide additional evidence on heterogeneous firm innovation intensity pattern from 2004 to 2009 by plotting the binned scatters between expenditure share of high-income consumers and innovation intensity proxied by claims-weighted patent count per expenditure of thousand U.S. dollars. We exclude those firm observations which have no sales records from high-income buyers. The number of bins is 20 as default. Each point represents a sample firm. The slope of the fitted line, along with its high statistical significance level proxied by t-statistics, indicates that firms targeting the rich buyers engage in more innovation activities.

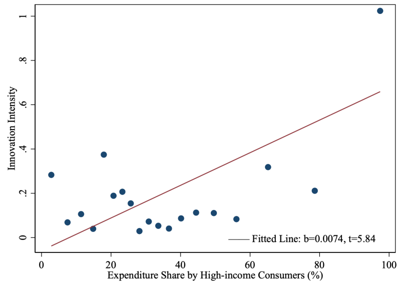
Figure A.13: Patent per Expenditure of Thousand U.S. Dollars and Expenditure Share of High-income Buyers, 2010-2016



(a) Year 2010 (b) Year 2011 (c) Year 2012



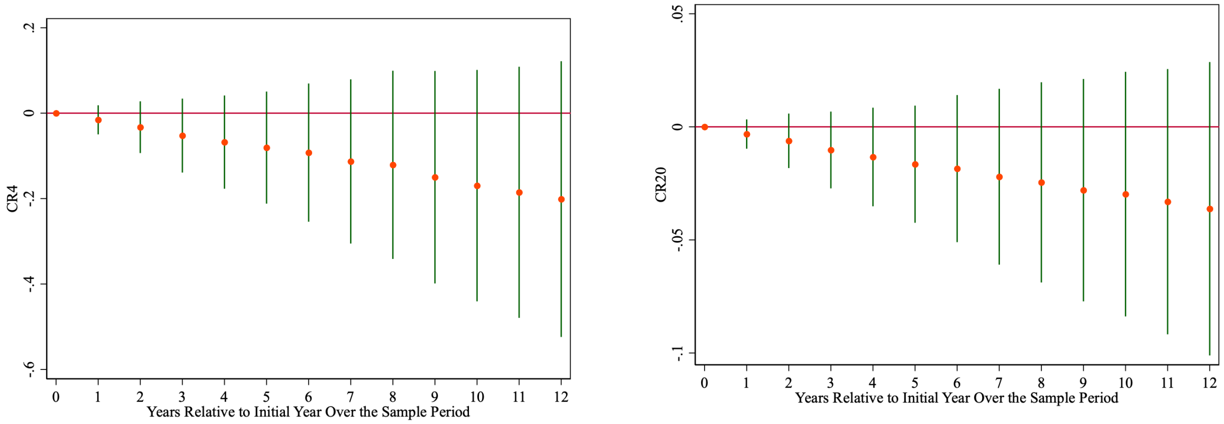
(d) Year 2013 (e) Year 2014 (f) Year 2015



(g) Year 2016

Notes. We provide additional evidence on heterogeneous firm innovation intensity pattern from 2010 to 2016 by plotting the binned scatters between expenditure share of high-income consumers and innovation intensity proxied by claims-weighted patent count per expenditure of thousand U.S. dollars. We exclude those firm observations which have no sales records from high-income buyers. The number of bins is 20 as default. Each point represents a sample firm. The slope of the fitted line, along with its high statistical significance level proxied by t-statistics, indicates that firms targeting the rich buyers engage in more innovation activities.

Figure A.14: Robustness: Evolution of Market Concentration over the Product Life Cycle, 2004-2016

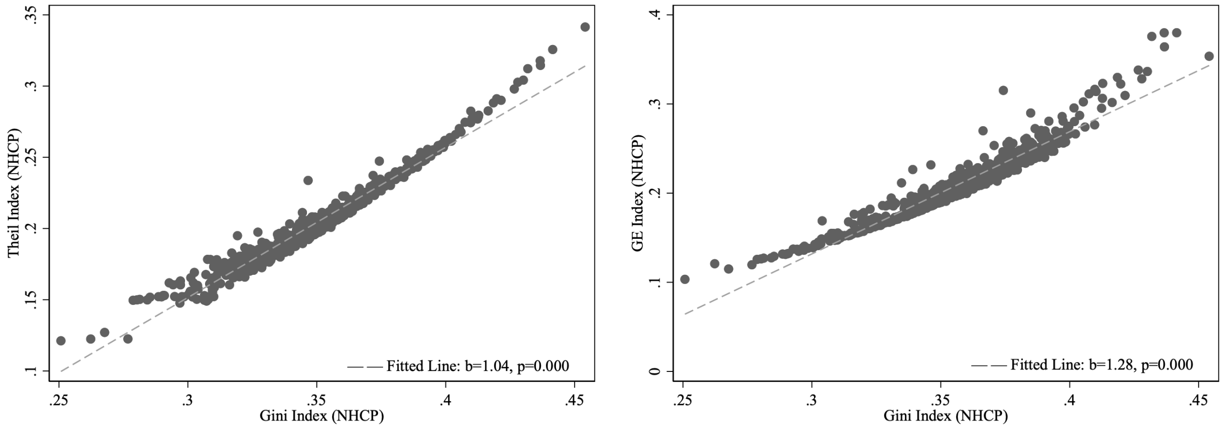


(a) Revenue Share of the Top Four Firms (CR4)

(b) Revenue Share of the Top Twenty Firms (CR20)

Notes. Each panel reports dynamic coefficients from regressions of market concentration measures on years since a product module's introduction, controlling for module and year fixed effects. Solid dots denote point estimates, and vertical lines indicate 95% confidence intervals based on robust standard errors clustered at the product-module level. Both CR4 and CR20 decline steadily as product markets mature, confirming that firm entry over the product cycle reduces concentration and strengthens competition.

Figure A.15: Correlation between Inequality Index

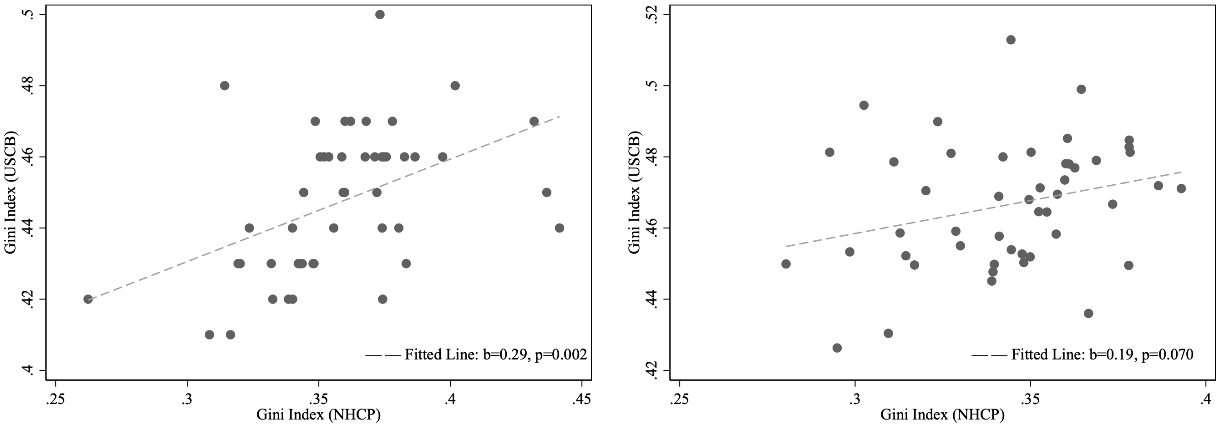


(a) Theil and Gini Index

(b) GE and Gini Index

Notes. Each point represents a U.S. state-year observation. The slope of the fitted line, along with its statistical significance level, indicates the explanatory power of each grouping variable.

Figure A.16: Validation of Inequality Index



(a) Year 2006

(b) Year 2016

Notes. Each point represents a U.S. state. The slope of the fitted line, along with its statistical significance level, indicates the explanatory power of each grouping variable.

Figure A.17: Distribution and Evolution of Gini Index

Notes. Figure shows the kernel density of the state-level Gini index in the United States for years 2004 and 2016, computed from the NHCP.

A.6 Table Appendix

Table A.1: Industry Distribution of Sample Firms (Top NAICS Four-Digit Sectors)

NAICS	Industry Description	Firm Count
4244	Grocery and Related Product Merchant Wholesalers	3504
5614	Business Support Services	2137
3121	Beverage Manufacturing	1975
3119	Other Food Manufacturing	1502
3118	Bakeries and Tortilla Manufacturing	1363
7225	Restaurants and Other Eating Places	999
4452	Specialty Food Retailers	919
3114	Fruit and Vegetable Preserving and Specialty Food Manufacturing	669
4249	Miscellaneous Nondurable Goods Merchant Wholesalers	614
4599	Other Miscellaneous Retailers	545
3399	Other Miscellaneous Manufacturing	481
4248	Beer, Wine, and Distilled Alcoholic Beverage Merchant Wholesalers	469
4239	Miscellaneous Durable Goods Merchant Wholesalers	466
3256	Soap, Cleaning Compound, and Toilet Preparation Manufacturing	464
3115	Dairy Product Manufacturing	441
5416	Management, Scientific, and Technical Consulting Services	410
4451	Grocery and Convenience Retailers	409
3254	Pharmaceutical and Medicine Manufacturing	393
3116	Animal Slaughtering and Processing	392
3113	Sugar and Confectionery Product Manufacturing	339
4242	Drugs and Druggists' Sundries Merchant Wholesalers	310
1113	Fruit and Tree Nut Farming	268
3261	Plastics Product Manufacturing	251
1119	Other Crop Farming	222
3112	Grain and Oilseed Milling	221
1121	Cattle Ranching and Farming	205
4232	Furniture and Home Furnishing Merchant Wholesalers	179
4234	Professional and Commercial Equipment and Supplies Merchant Wholesalers	178
5511	Management of Companies and Enterprises	177
4238	Machinery, Equipment, and Supplies Merchant Wholesalers	168
3222	Converted Paper Product Manufacturing	158
4243	Apparel, Piece Goods, and Notions Merchant Wholesalers	154
5131	Newspaper, Periodical, Book, and Directory Publishers	146
5121	Motion Picture and Video Industries	146
5239	Other Financial Investment Activities	146
4453	Beer, Wine, and Liquor Retailers	141
4885	Freight Transportation Arrangement	138
4561	Health and Personal Care Retailers	134
1112	Vegetable and Melon Farming	130
3111	Animal Food Manufacturing	129
4594	Office Supplies, Stationery, and Gift Retailers	129
3391	Medical Equipment and Supplies Manufacturing	128
4236	Household Appliances and Electrical and Electronic Goods Merchant Wholesalers	119
4591	Sporting Goods, Hobby, and Musical Instrument Retailers	116
1151	Support Activities for Crop Production	115
4581	Clothing and Clothing Accessories Retailers	113
4492	Electronics and Appliance Retailers	112
1114	Greenhouse, Nursery, and Floriculture Production	108
4491	Furniture and Home Furnishings Retailers	104
3152	Cut and Sew Apparel Manufacturing	103

Notes. The table reports the distribution of sample firms across the top NAICS four-digit industries (2022 classification). To conserve space, only industries with more than 100 firms are shown. Consumer-oriented industries such as grocery wholesaling, food manufacturing, and beverage production dominate, reflecting the prevalence of firms producing everyday goods.

Table A.2: Patent per Expenditure of Thousand U.S. Dollars Across Firms

Year	Cut-off: 40th percentile		Cut-off: 50th percentile		Cut-off: 60th percentile	
	Rich Firms	Poor Firms	Rich Firms	Poor Firms	Rich Firms	Poor Firms
2004	0.0176	0.0038	0.0204	0.0037	0.0244	0.0038
	(0.4157)	(0.0547)	(0.4542)	(0.0586)	(0.5072)	(0.0569)
2005	0.0232	0.0038	0.0270	0.0040	0.0328	0.0039
	(0.5284)	(0.0460)	(0.5765)	(0.0657)	(0.6429)	(0.0698)
2006	0.0239	0.0033	0.0278	0.0035	0.0339	0.0035
	(0.7504)	(0.0311)	(0.8216)	(0.0369)	(0.9184)	(0.0363)
2007	0.0234	0.0138	0.0241	0.0151	0.0282	0.0138
	(0.4997)	(0.4514)	(0.4760)	(0.4858)	(0.5291)	(0.4459)
2008	0.0335	0.0190	0.0395	0.0160	0.0467	0.0151
	(0.7965)	(0.9233)	(0.8716)	(0.8266)	(0.9695)	(0.7588)
2009	0.0372	0.0040	0.0435	0.0045	0.0525	0.0049
	(1.0572)	(0.0381)	(1.1576)	(0.0482)	(1.2901)	(0.0940)
2010	0.0284	0.0092	0.0327	0.0088	0.0396	0.0082
	(0.6785)	(0.1853)	(0.7421)	(0.1705)	(0.8290)	(0.1576)
2011	0.0583	0.0061	0.0688	0.0061	0.0844	0.0061
	(2.5254)	(0.0662)	(2.7660)	(0.0797)	(3.0920)	(0.0795)
2012	0.0198	0.0102	0.0202	0.0116	0.0232	0.0111
	(0.3828)	(0.2743)	(0.3250)	(0.3611)	(0.3524)	(0.3374)
2013	0.0605	0.0076	0.0719	0.0068	0.0885	0.0065
	(3.4953)	(0.1062)	(3.8287)	(0.0965)	(4.2802)	(0.0985)
2014	0.0247	0.0108	0.0286	0.0097	0.0348	0.0087
	(0.4007)	(0.3985)	(0.4377)	(0.3579)	(0.4888)	(0.3271)
2015	0.0164	0.0099	0.0187	0.0090	0.0210	0.0091
	(0.2868)	(0.1983)	(0.3129)	(0.1795)	(0.3406)	(0.1764)
2016	0.0206	0.0130	0.0241	0.0110	0.0292	0.0098
	(0.3558)	(0.3013)	(0.3893)	(0.2701)	(0.4347)	(0.2470)

Notes. This table reports summary statistics on total patent count per expenditure of thousand U.S. dollars across firms over the sample period (2004-2016). Rich firms refer to those firms with expenditure share of high-income consumers above the cut-off. The definition of firms targeting the poor is analogous to that of firms targeting the rich. We exclude those firm observations which have no sales records from high-income buyers. To ensure robustness of our results, we report summary statistics of 40th, 50th and 60th percentile cut-offs, respectively. All of these grouping criteria depict a highly similar picture: firms targeting the high-income buyers are inclined to innovate more. In parentheses are standard deviations of innovation intensity.

Table A.3: Claims-weighted Patent per Expenditure of Thousand U.S. Dollars Across Firms

Year	Cut-off: 40th percentile		Cut-off: 50th percentile		Cut-off: 60th percentile	
	Rich Firms	Poor Firms	Rich Firms	Poor Firms	Rich Firms	Poor Firms
2004	0.2373	0.0655	0.2727	0.0646	0.3275	0.0627
	(4.3846)	(0.9568)	(4.7681)	(1.0294)	(5.3249)	(0.9587)
2005	0.3180	0.0569	0.3663	0.0609	0.4418	0.0614
	(7.0894)	(0.8376)	(7.7064)	(1.2132)	(8.5637)	(1.3447)
2006	0.3171	0.0479	0.3681	0.0507	0.4467	0.0512
	(10.1090)	(0.4898)	(11.0673)	(0.5620)	(12.3704)	(0.5546)
2007	0.3762	0.2312	0.4155	0.2210	0.4904	0.2034
	(8.4726)	(9.2434)	(9.1327)	(8.4309)	(10.1504)	(7.7484)
2008	0.4230	0.3324	0.5005	0.2730	0.5857	0.2542
	(7.2207)	(18.4206)	(7.9024)	(16.4795)	(8.7528)	(15.0747)
2009	0.4729	0.0553	0.5498	0.0621	0.6459	0.0793
	(15.1965)	(0.6148)	(16.6379)	(0.7660)	(18.4171)	(2.2442)
2010	0.4160	0.1480	0.4811	0.1365	0.5851	0.1247
	(11.1684)	(3.2472)	(12.2177)	(2.9702)	(13.6513)	(2.7339)
2011	0.7662	0.0852	0.9056	0.0821	1.1081	0.0843
	(37.9463)	(1.11998)	(41.5638)	(1.1803)	(46.4627)	(1.1898)
2012	0.2856	0.1396	0.2769	0.1775	0.3050	0.1753
	(6.4450)	(3.6982)	(4.8495)	(6.1053)	(4.9764)	(5.8436)
2013	1.0590	0.1006	1.2624	0.0888	1.5606	0.0856
	(65.6809)	(1.5705)	(71.9456)	(1.4213)	(80.4343)	(1.4155)
2014	0.3528	0.1694	0.4066	0.1523	0.4941	0.1364
	(6.4413)	(7.2725)	(7.0208)	(6.5416)	(7.8395)	(5.9785)
2015	0.2077	0.1452	0.2352	0.1301	0.2616	0.1300
	(3.3560)	(3.4660)	(3.6590)	(3.1199)	(3.8626)	(3.0529)
2016	0.2639	0.1778	0.3065	0.1524	0.3714	0.1348
	(4.9505)	(4.0705)	(5.4137)	(3.6535)	(6.0483)	(3.3390)

Notes. This table reports summary statistics on claims-weighted patent count per expenditure of thousand U.S. dollars across firms over the sample period (2004-2016). Rich firms refer to those firms with expenditure share of high-income consumers above the cut-off. The definition of firms targeting the poor is analogous to that of firms targeting the rich. We exclude those firm observations which have no sales records from high-income buyers. To ensure robustness of our results, we report summary statistics of 40th, 50th and 60th percentile cut-offs, respectively. All of these grouping criteria depict a highly similar picture: firms targeting the high-income buyers are inclined to innovate more. In parentheses are standard deviations of innovation intensity.

Table A.4: Income Inequality Exposure and Firm Innovation (Theil Index)

	Baseline and Robustness						Heterogeneity	
Dep var. N_{ft}^{Pat}	(1) Full Sample	(2) No Entry Year	(3) Poor and Rich	(4) Innovative Firm	(5) Big Firm	(6) Surviving Frim	(7) Targeting Poor	(8) Targeting Rich
A. Dependent variable: Patent applications								
$Shock_{ft}^{ineq}$	-0.006**	-0.007*	-0.006**	-0.006*	-0.006*	-0.006**	-0.006**	-0.002
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$Shock_{ft}^{inc}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.001
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Firm FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Observations	428,544	384,656	270,604	306,344	215,166	283,010	214,802	214,063
Adjusted R^2	0.672	0.681	0.675	0.655	0.682	0.680	0.678	0.689

	Baseline and Robustness						Heterogeneity	
Dep var. N_{ft}^{Pat}	(1) Full Sample	(2) No Entry Year	(3) Poor and Rich	(4) Innovative Firm	(5) Big Firm	(6) Surviving Frim	(7) Targeting Poor	(8) Targeting Rich
B. Dependent variable: Claim-weighted patent applications								
$Shock_{ft}^{ineq}$	-0.010**	-0.011**	-0.010**	-0.009**	-0.009**	-0.010**	-0.010**	-0.007
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$Shock_{ft}^{inc}$	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	-0.002
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Firm FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Observations	428,544	384,656	270,604	306,344	215,166	283,010	214,802	214,063
Adjusted R^2	0.517	0.526	0.520	0.484	0.531	0.525	0.526	0.532

Notes. This table reports firm-year regressions of innovation on exposure to income-inequality shocks. In Panel A, the dependent variable is IHS($Patents_{ft}$), where $Patents_{ft}$ is the number of patent applications filed by firm f in year t (PatentsView/USPTO). In Panel B, the dependent variable is IHS($Claim\text{-}weighted\ Patents_{ft}$), where patents are weighted by the number of claims. The main regressor, $Shock_{ft}^{ineq}$, is the firm-specific inequality exposure in equation (2): a weighted average of predicted state-year inequality, with weights given by the firm's baseline expenditure shares across states in its entry year. We also control for $Shock_{ft}^{inc}$, constructed analogously by replacing predicted inequality with a predicted state-year measure of mean household income; this control separates changes in dispersion from shifts in income levels. Columns (1)–(6) report baseline and robustness samples: (2) excludes each firm's entry year; (3) restricts to firms with positive exposure to both low- and high-income consumers; (4) restricts to firms that patent at least once during 2004–2016; (5) restricts to firms with above-median revenues; and (6) restricts to firms observed in all sample years (balanced panel). Columns (7)–(8) split firms by sales orientation: *Targeting Poor* (*Targeting Rich*) indicates firms with above-median (below-median) expenditure shares from low-income consumers, computed over the sample period. All specifications include firm and year fixed effects. Controls include $\ln(1 + Revenues_{ft})$, the firm's revenue share (market-power proxy), $\ln(1 + \#Modules_{ft})$, $\ln(1 + \#States_{ft})$, and firm age. Standard errors (in parentheses) are clustered at the firm level. For readability, $Shock_{ft}^{ineq}$ and $Shock_{ft}^{inc}$ are multiplied by 1,000. ***, **, and * denote significance at the 1%, 5%, and 10% levels, respectively.

Table A.5: Income Inequality Exposure and Firm Innovation (GE Index)

	Baseline and Robustness						Heterogeneity	
Dep var. N_{ft}^{Pat}	(1) Full Sample	(2) No Entry Year	(3) Poor and Rich	(4) Innovative Firm	(5) Big Firm	(6) Surviving Frim	(7) Targeting Poor	(8) Targeting Rich
A. Dependent variable: Patent applications								
$Shock_{ft}^{ineq}$	-0.004**	-0.004*	-0.004*	-0.004*	-0.004*	-0.004*	-0.004**	-0.001
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$Shock_{ft}^{inc}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.001
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Firm FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Observations	428,544	384,656	270,305	306,344	215,166	283,010	214,802	214,063
Adjusted R^2	0.672	0.681	0.676	0.655	0.682	0.680	0.678	0.689

	Baseline and Robustness						Heterogeneity	
Dep var. N_{ft}^{Pat}	(1) Full Sample	(2) No Entry Year	(3) Poor and Rich	(4) Innovative Firm	(5) Big Firm	(6) Surviving Frim	(7) Targeting Poor	(8) Targeting Rich
B. Dependent variable: Claim-weighted patent applications								
$Shock_{ft}^{ineq}$	-0.006**	-0.007**	-0.006**	-0.006*	-0.006**	-0.006**	-0.006**	-0.004
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$Shock_{ft}^{inc}$	-0.000	-0.001	-0.000	-0.001	-0.000	-0.000	-0.000	-0.002
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Firm FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Observations	428,544	384,656	270,305	306,344	215,166	283,010	214,802	214,063
Adjusted R^2	0.517	0.526	0.521	0.484	0.531	0.525	0.526	0.532

Notes. This table reports firm-year regressions of innovation on exposure to income-inequality shocks. In Panel A, the dependent variable is IHS($Patents_{ft}$), where $Patents_{ft}$ is the number of patent applications filed by firm f in year t (PatentsView/USPTO). In Panel B, the dependent variable is IHS($\textit{Claim-weighted Patents}_{ft}$), where patents are weighted by the number of claims. The main regressor, $Shock_{ft}^{ineq}$, is the firm-specific inequality exposure in equation (2): a weighted average of predicted state-year inequality, with weights given by the firm's baseline expenditure shares across states in its entry year. We also control for $Shock_{ft}^{inc}$, constructed analogously by replacing predicted inequality with a predicted state-year measure of mean household income; this control separates changes in dispersion from shifts in income levels. Columns (1)–(6) report baseline and robustness samples: (2) excludes each firm's entry year; (3) restricts to firms with positive exposure to both low- and high-income consumers; (4) restricts to firms that patent at least once during 2004–2016; (5) restricts to firms with above-median revenues; and (6) restricts to firms observed in all sample years (balanced panel). Columns (7)–(8) split firms by sales orientation: *Targeting Poor* (*Targeting Rich*) indicates firms with above-median (below-median) expenditure shares from low-income consumers, computed over the sample period. All specifications include firm and year fixed effects. Controls include $\ln(1 + Revenues_{ft})$, the firm's revenue share (market-power proxy), $\ln(1 + \#Modules_{ft})$, $\ln(1 + \#States_{ft})$, and firm age. Standard errors (in parentheses) are clustered at the firm level. For readability, $Shock_{ft}^{ineq}$ and $Shock_{ft}^{inc}$ are multiplied by 1,000. ***, **, and * denote significance at the 1%, 5%, and 10% levels, respectively.

A.7 Income Inequality and Firm Innovation

To investigate how income inequality influences firm innovation, we estimate the following regression:

$$N_{ft}^{Pat} = \beta \underbrace{\sum_{s} \lambda_{sft_0} \widehat{Gini}_{st}}_{Shock_{ft}^{ineq}} + \delta \underbrace{\sum_{s} \lambda_{sft_0} \ln(\widehat{Income}_{st})}_{Shock_{ft}^{inc}} + \mathbf{X_{ft}}\gamma' + \mu_f + \mu_t + \epsilon_{ft}, \tag{A.1}$$

where f indexes firm, t indexes year, and s indexes U.S. state. The initial share of firm f's sales in state s in its entry year t_0 is defined as $\lambda_{sft_0} = E_{sft_0}/\sum_f E_{sft_0}$, where E denotes expenditure. Following Boyce et al. (2016), state-year inequality is defined in the form of Gini index as

$$Gini = 1 + \frac{1}{n} - \frac{2}{n^2 \cdot Mean(Income_i)} \cdot \sum_{i=1}^{n} [(n - i + 1) \cdot Income_i].$$

$Income$ is household income. n is the number of households, indexed in non-decreasing order. The Gini coefficient lies in the interval between zero and one, with higher values denoting greater inequality. Our instrumental variable strategy adheres to the identification logic developed by Doerr et al. (2024), interacting each U.S. state's initial inequality level with the subsequent national evolution of inequality:

$$\widehat{Gini}_{st} = Gini_{s,2004} \times \frac{1}{N-1} \sum_{j \neq s}^{N} \frac{Gini_{jt}}{Gini_{j,2004}}. \tag{A.2}$$

The constructed measure, $\widehat{Gini}_{st}$, captures the counterfactual evolution of inequality in state s if it had tracked the national trend, scaled by its initial inequality level. We then use this predicted series as an instrument for the observed state-level Gini index between 2004 and 2016.

To ensure the mean-preserving property that income distribution change has, we control firm's exposure to household income–which also eliminates the potential effect of income level of the market on firm innovation. The shock of income, $Shock_{ft}^{inc}$, is constructed in a similar way as $Shock_{ft}^{ineq}$. We control for firm characteristics that capture scale, market position, and scope: firm revenue, to distinguish innovation intensity from firm size; the firm's revenue share within its markets, as a proxy for market power; the number of modules supplied, to capture product scope; the number of states served, to capture geographic scope; and firm age, measured as years since first entry during 2004–2016. We control for unobservable firm- and year-level shocks by including firm and year fixed effects. All robust standard errors are clustered at the firm level.

To ensure the robustness of our main results, we consider several alternative income inequality measures. The first one is the Theil index, which is also sensitive to the middle range of the distribution, like the Gini index. The Theil index is calculated as follows:

$$Theil = \sum_{i=1}^{n} \omega_i \cdot \frac{Income_i}{Mean(Income_i)} \cdot \ln\left(\frac{Income_i}{Mean(Income_i)}\right),$$

where ω_i is the population share for household i. We finally calculate a Generalized Entropy (GE) family of inequality measure, which is more sensitive to changes in the upper range of the distribution:

$$GE = \frac{1}{2}\sum_{i=1}^{n} \omega_i \cdot \left(\frac{Income_i}{Mean(Income_i)}\right)^2 - \frac{1}{2}.$$

These inequality measures are all calculated using individual household income data from the NielsenIQ Homescan Consumer Panel (NHCP).

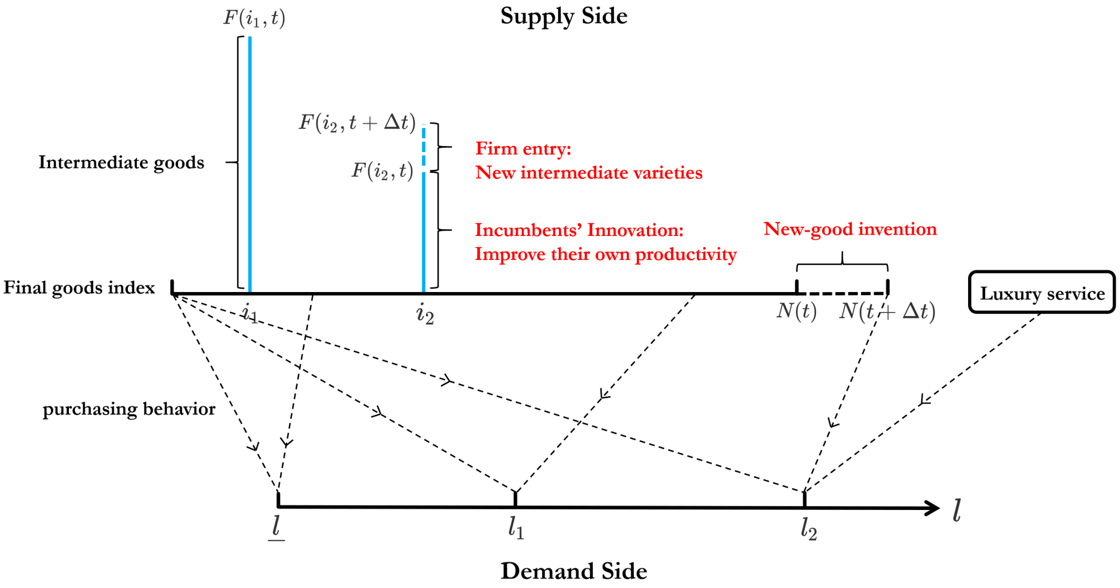
B Appendix: Baseline Model

In this appendix, we provide additional materials that clarify the overall structure of the model and present the detailed derivations. The model environment is the same as in the main text.

B.1 Summary of the Model Structure

Figure B.1 summarizes the structure of the model. On the demand side, a continuum of heterogeneous households, ordered by their endowment l, consume a hierarchy of goods whose affordability rises with income. Their non-homothetic preferences generate an endogenous demand ladder: richer households purchase goods with higher index i, and only the top-income group consumes the luxury service at the upper end of the hierarchy.

Figure B.1: Summary of the Model Structure



On the supply side, each final good is assembled by a final-goods producer that aggregates a continuum of intermediate varieties. Economic growth arises from multiple layers of innovation. At the frontier, inventors introduce new categories of final consumption goods. Within each sector, incumbent firms conduct R&D to improve productivity, while potential entrants decide whether to enter based on expected firm value. Productivity improvements and firm entry jointly expand sectoral output, lower prices, and extend market access to lower-income households, gradually transforming luxury goods into necessities.

In equilibrium, rising productivity and wages reshape the demand distribution across goods, guiding the direction of the next wave of innovation. The interaction between hierarchical demand, firm dynamics, and innovation thus endogenously generates a BGP in which goods sequentially diffuse down the income distribution, linking inequality, innovation, and long-run economic growth.

B.2 Derivations and Proofs for Section 3

B.2.1 Derivations for Section 3.2

We derive the profit function of intermediate-good producers, $\Pi_i(j,t)$, and then establish the expressions for sectoral output $Y(i,t)$ and labor demand $L(i,t)$.

First, we solve the optimization problem of the representative final-good producer:

$$\Pi(i,t) = \text{argmax}_{\{M_i(j,t)\}} \left\{ \frac{P(i,t)}{1-\beta} \int_0^{F(i,t)} M_i(j,t)^{1-\beta} dj - \int_0^{F(i,t)} Q_i(j,t) M_i(j,t) dj \right\},$$

the first-order condition implies the inverse demand for each intermediate variety is

$$P(i,t) M_i(j,t)^{-\beta} = Q_i(j,t).$$

Each intermediate-good producer operates under monopolistic competition and chooses $M_i(j,t)$ to maximize profits

$$\Pi_i(j,t) = \text{argmax}_{M_i(j,t)} \Big\{ P(i,t) M_i(j,t)^{1-\beta} - W(t) \cdot \frac{M_i(j,t)}{\tilde{A}_i(j,t)^{\theta} \bar{A}(t)^{1-\theta}} \Big\}.$$

The first-order condition yields

$$M_i(j,t) \;=\; \left[\frac{W(t)}{(1-\beta) P(i,t) \tilde{A}_i(j,t)^{\theta} \bar{A}(t)^{1-\theta}} \right]^{-\frac{1}{\beta}},$$

and

$$Q_i(j,t) \;=\; \frac{W(t)}{(1-\beta) \tilde{A}_i(j,t)^{\theta} \bar{A}(t)^{1-\theta}},$$

Substituting the expression of $M_i(j,t)$ into the profit function of the intermediate-good producer gives

$$\begin{aligned} \Pi_i(j,t) &= \frac{\beta}{1-\beta} \frac{W(t)}{\tilde{A}_i(j,t)^{\theta} \bar{A}(t)^{1-\theta}} \left[\frac{W(t)}{(1-\beta) P(i,t) \tilde{A}_i(j,t)^{\theta} \bar{A}(t)^{1-\theta}} \right]^{-\frac{1}{\beta}} \\ &= \beta (1-\beta)^{\frac{1-\beta}{\beta}} \left[\frac{P(i,t)^{\frac{1}{1-\beta}}}{W(t)} \bar{A}(t)^{1-\theta} \tilde{A}_i(j,t)^{\theta} \right]^{\frac{1-\beta}{\beta}}, \end{aligned} \tag{B.1}$$

which corresponds to equation (9) in the main text. Also, we obtain the profit of the final-good producer

$$\Pi(i,t) \;=\; \frac{P(i,t)}{1-\beta} \int_0^{F(i,t)} M_i(j,t)^{1-\beta} dj - \int_0^{F(i,t)} Q_i(j,t) M_i(j,t) dj$$

$$
\begin{aligned}
&= \frac{P(i,t)}{1-\beta}\int_0^{F(i,t)} \left[\frac{W(t)}{(1-\beta)P(i,t)\tilde{A}_i(j,t)^\theta \bar{A}(t)^{1-\theta}}\right]^{-\frac{1-\beta}{\beta}} dj \\
&\quad - \int_0^{F(i,t)} \frac{W(t)}{(1-\beta)\tilde{A}_i(j,t)^\theta\bar{A}(t)^{1-\theta}}\left[\frac{W(t)}{(1-\beta)P(i,t)\tilde{A}_i(j,t)^\theta\bar{A}(t)^{1-\theta}}\right]^{-\frac{1}{\beta}} dj \\
&= \frac{\beta}{1-\beta}\int_0^{F(i,t)} (1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{P(i,t)^{\frac{1}{1-\beta}}}{W(t)}\bar{A}(t)^{1-\theta}\tilde{A}_i(j,t)^\theta\right]^{\frac{1-\beta}{\beta}} dj \\
&= \frac{1}{1-\beta}\int_0^{F(i,t)}\Pi_i(j,t)dj. \qquad \text{(B.2)}
\end{aligned}
$$

The last equation follows from equation (B.1), and it states that the profit of the final-good producer equals the total profit of the intermediate-good producer times $\frac{1}{1-\beta}$.

Next, aggregate across all $j \in [0, F(i,t)]$ to derive the sectoral output

$$
\begin{aligned}
Y(i,t) &= \frac{1}{1-\beta}\int_0^{F(i,t)} M_i(j,t)^{1-\beta} dj \\
&= (1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{W(t)}{P(i,t)}\right]^{-\frac{1-\beta}{\beta}} \bar{A}(t)^{(1-\theta)\frac{1-\beta}{\beta}}\int_0^{F(i,t)} \tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}} dj \\
&= (1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{W(t)}{P(i,t)}\right]^{-\frac{1-\beta}{\beta}} B(i,t),
\end{aligned}
$$

where the last equality follows from the definition of $B(i,t)$ in equation (10).

Finally, total labor demand in sector i is

$$
\begin{aligned}
L(i,t) &= \int_0^{F(i,t)} L_i(j,t)dj \\
&= \left[\frac{W(t)}{(1-\beta)P(i,t)}\right]^{-\frac{1}{\beta}} \bar{A}(t)^{(1-\theta)\frac{1-\beta}{\beta}}\int_0^{F(i,t)} \tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}} dj \\
&= \left[\frac{W(t)}{(1-\beta)P(i,t)}\right]^{-\frac{1}{\beta}} B(i,t),
\end{aligned}
$$

which corresponds to equations (12) in the main text.

At last, we discuss the profit allocation in the production sector. Substitute the definition of $B(i,t)$ in equation (B.2)

$$
\begin{aligned}
\Pi(i,t) &= \frac{\beta}{1-\beta}\int_0^{F(i,t)} (1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{P(i,t)^{\frac{1}{1-\beta}}}{W(t)}\bar{A}(t)^{1-\theta}\tilde{A}_i(j,t)^\theta\right]^{\frac{1-\beta}{\beta}} dj \\
&= \beta(1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{W(t)}{P(i,t)^{\frac{1}{1-\beta}}}\right]^{-\frac{1-\beta}{\beta}} B(i,t) \\
&= \beta P(i,t)Y(i,t).
\end{aligned}
$$

This relation shows that a share β of output in each sector accrues to the final-good producer. From equation (B.2), we also know that among the remaining share $1-\beta$, a fraction β accrues to intermediate-good producers, i.e.

$$\frac{\Pi(i,t)}{P(i,t)Y(i,t)} = \beta \quad \text{and} \quad \frac{\int_0^{F(i,t)} \Pi_i(j,t)dj}{P(i,t)Y(i,t)} = \beta(1-\beta). \tag{B.3}$$

B.2.2 Derivations and Proofs for Section 3.3

The main purpose of this appendix is to introduce Lemma B.1. Lemma B.1 shows that the value function is separable in firm-specific productivity and aggregate variables, implying that heterogeneity among firms within a sector can be summarized by the common sectoral component $V_i(t)$. As a result, the R&D intensity $z_i(t)$ is identical across firms in the same sector and depends solely on sector-level fundamentals. This property substantially simplifies the characterization of equilibrium innovation dynamics.

Lemma B.1 *Under Assumption 1, the value function $\tilde{V}_i(A,t)$ is linear on $A^{\theta\frac{1-\beta}{\beta}}$ and can be expressed as*

$$\tilde{V}_i(A,t) = A^{\theta\frac{1-\beta}{\beta}} V_i(t), \tag{B.4}$$

where $V_i(t)$ only depends on sector-level or aggregate-level variable. The associated R&D intensity is given by

$$z_i(j,t) = \psi \left[\frac{W(t)}{\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right) V_i(t) \cdot \psi\alpha\bar{A}(t)^{1+\theta\frac{1-\beta}{\beta}}} \right]^{\frac{\alpha}{\alpha-1}} \equiv z_i(t). \tag{B.5}$$

Proof of Lemma B.1:

To solve the HJB equation (14), we adopt the standard guess-and-verify approach. Conjecture that the value function takes the form:

$$\tilde{V}_i(A,t) = A^{\theta\frac{1-\beta}{\beta}} V_i(t). \tag{B.6}$$

Substituting equation (B.6) into the incumbent firm's HJB equation (14) yields

$$\begin{aligned} rA^{\theta\frac{1-\beta}{\beta}}V_i(t) - A^{\theta\frac{1-\beta}{\beta}}\dot{V}_i(t) = \max_{L_i^I(j,t)} \beta(1-\beta)^{\frac{1-\beta}{\beta}} &\left[\frac{P(i,t)^{\frac{1}{1-\beta}}}{W(t)} \bar{A}(t)^{1-\theta} A^{\theta} \right]^{\frac{1-\beta}{\beta}} \\ &- W(t)L_i^I(j,t) + z_i(j,t)\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right) A^{\theta\frac{1-\beta}{\beta}} V_i(t) - \delta A^{\theta\frac{1-\beta}{\beta}} V_i(t). \end{aligned} \tag{B.7}$$

Using $z_i(j,t) = \psi\bar{A}(t)^{\xi}\tilde{A}_i(j,t)^{-\iota}\left(L_i^I(j,t)\right)^{\alpha}$, where we have denoted

$$\xi = \alpha\left(1 + \theta\frac{1-\beta}{\beta}\right) \tag{B.8}$$

to simplify expression, together with Assumption 1 that $\iota = \alpha\theta^{\frac{1-\beta}{\beta}}$, the first-order condition with respect to $L_i^I(t)$ yields

$$L_i^I(j,t) = A^{\theta^{\frac{1-\beta}{\beta}}} \left[\frac{W(t)}{\left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1\right) V_i(t)\psi\alpha\bar{A}(t)^{\xi}}\right]^{\frac{1}{\alpha-1}}. \tag{B.9}$$

Substituting into $z_i(j,t)$ gives the optimal innovation effort

$$z_i(j,t) = \psi \left[\frac{W(t)}{\left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1\right) V_i(t)\psi\alpha\bar{A}(t)^{\frac{\xi}{\alpha}}}\right]^{\frac{\alpha}{\alpha-1}}.$$

Plugging $L_i^I(j,t)$ and $z_i(j,t)$ into equation (B.7), and dividing both sides by $A^{\theta^{\frac{1-\beta}{\beta}}}$, we obtain

$$\begin{aligned} rV_i(t) - \dot{V}_i(t) = {} & \beta(1-\beta)^{\frac{1-\beta}{\beta}} \left[\frac{P(i,t)^{\frac{1}{1-\beta}}}{W(t)}\bar{A}(t)^{1-\theta}\right]^{\frac{1-\beta}{\beta}} \\ & - W(t)\left[\frac{W(t)}{\left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1\right) V_i(t)\psi\alpha\bar{A}(t)^{\xi}}\right]^{\frac{1}{\alpha-1}} \\ & + \psi\left[\frac{W(t)}{\left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1\right) V_i(t)\psi\alpha\bar{A}(t)^{\xi/\alpha}}\right]^{\frac{\alpha}{\alpha-1}} \left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1\right) V_i(t) - \delta V_i(t). \end{aligned} \tag{B.10}$$

Crucially, the resulting equation is independent of firm-level productivity A, confirming the conjecture in equation (B.6). Hence, the value function $\tilde{V}_i(A,t)$ is linear in $A^{\theta^{\frac{1-\beta}{\beta}}}$, and $V_i(t)$ depends only on sectoral and aggregate variables such as r, $P(i,t)$, $W(t)$, and $\bar{A}(t)$. The associated R&D intensity is given by equation (B.5). This completes the proof. ■

B.2.3 Labor for Firm Entry

In this appendix, we discuss a technical issue related to the labor demand for firm entry. We begin by characterizing the dynamics of the number of firms in sector i, denoted by $F(i,t)$. In Lemma B.2, we show that $F(i,t)$ is a càdlàg process and may exist discontinuity only when $i = N(t)$. The proof of Lemma B.2 is somewhat technical and relies on several results established later for the BGP. Readers may therefore find it helpful to first review the statement and main intuition of the lemma, and then return to the detailed proof after reading the subsequent results.

Lemma B.2 *Fix a sector k, and let t^* denote the time such that $k = N(t^*)$. In the stationary equilibrium, the firm mass $F(k,t)$ satisfies the following properties: (i) for $t < t^*$, $F(k,t) = 0$; (ii) at $t = t^*$, $F(k,t) = F_1 > 0$, where F_1 is a constant; (iii) for $t > t^*$, $F(k,t)$ is continuous and* *differentiable in t.*

Proof of Lemma B.2:

Since $N(t)$ is strictly increasing, sector k has not yet been invented for all $t < t^*$. Therefore,

$$F(k,t) = 0 \quad \text{for all } t < t^*.$$

At time $t = t^*$, Lemma B.5 implies that the demand for final good k is strictly positive and given by $1 - \mathcal{G}(\bar{l}) > 0$. Using equation (11), sectoral output satisfies

$$\begin{aligned} 1 - \mathcal{G}(\bar{l}) &= (1-\beta)^{\frac{1-2\beta}{\beta}} \left[\frac{W(t^*)}{P(k,t^*)}\right]^{-\frac{1-\beta}{\beta}} B(k,t^*) \\ &= (1-\beta)^{\frac{1-2\beta}{\beta}} \left[\frac{W(t^*)}{P(k,t^*)}\right]^{-\frac{1-\beta}{\beta}} \bar{A}(t^*)^{(1-\theta)\frac{1-\beta}{\beta}} \int_0^{F(k,t^*)} \tilde{A}_k(j,t^*)^{\theta\frac{1-\beta}{\beta}} dj \\ &= (1-\beta)^{\frac{1-2\beta}{\beta}} \left[\frac{W(t^*)}{\bar{A}(t^*)P(k,t^*)}\right]^{-\frac{1-\beta}{\beta}} \zeta^{\theta\frac{1-\beta}{\beta}} F(k,t^*). \end{aligned}$$

Since the left-hand side is strictly positive, it follows that

$$F(k,t^*) > 0.$$

Furthermore, by the time-translation invariance property established in Lemma B.10, firm mass depends only on the normalized index $\tilde{i} = i/N(t)$. At time t^*, we have $\tilde{i} = k/N(t^*) = 1$ by construction. Therefore,

$$F(k,t^*) = F_1,$$

where F_1 is a constant.

Finally, for $t > t^*$, entrants solve

$$\max_{L_k^E(t)} \left\{ e(k,t) \cdot \mathbb{E}_t \left[A^{\theta\frac{1-\beta}{\beta}} V_k(t) \right] - W(t) L_k^E(t) \right\},$$

where $e(k,t) = \psi_e \bar{A}(t) L_k^E(t)$.The first-order condition implies

$$\begin{aligned} W(t) &= V_k(t)\psi_e \bar{A}(t) \mathbb{E}_t \left[A^{\theta\frac{1-\beta}{\beta}} \right] \\ &= V_k(t)\psi_e \bar{A}(t) \frac{B(k,t)}{\bar{A}(t)^{(1-\theta)\frac{1-\beta}{\beta}} F(k,t)}. \end{aligned} \tag{B.11}$$

Rearranging yields

$$F(k,t) = V_k(t)\psi_e \bar{A}(t) \frac{B(k,t)}{\bar{A}(t)^{(1-\theta)\frac{1-\beta}{\beta}} W(t)}. \tag{B.12}$$

Since $W(t)$ and $\bar{A}(t)$ grow at constant rates, they are continuous and differentiable in t. Equation (B.32) implies that $V_k(t)$ is continuous and differentiable, which in turn implies from equation (B.33) that $P(k,t)$ is continuous and differentiable. Equation (11), together with Lemma B.8, implies that both $Y(k,t)$ and $B(k,t)$ are continuous and differentiable. Therefore, equation (B.12) implies that $F(k,t)$ is continuous and differentiable in t for all $t > t^*$. ■

To accommodate the possible discontinuities in firm creation characterized in Lemma B.2, we define the cumulative entry–labor process $L^{AE}(i,t)$ as the total amount of labor devoted to firm entry in sector i up to time t,

$$L^{AE}(i,t) = \int_0^t L^E(i,\tau)\, d\tau.$$

This cumulative formulation allows entry to occur both through continuous flows and through instantaneous jumps, while preserving bounded variation. Accordingly, the law of motion for firm mass can be written in the general differential form

$$dF(i,t) = \psi_e \bar{A}(t)\, dL^{AE}(i,t) - \delta F(i,t)\, dt, \tag{B.13}$$

where differentials with respect to L^{AE} are interpreted in the Lebesgue–Stieltjes sense. When entry is continuous, this expression reduces to the flow representation $\dot{F}(i,t) = e(i,t) - \delta F(i,t)$ used in the main text. Aggregating over sectors, define the total mass of intermediate-good firms at time t as

$$\bar{F}(t) \equiv \int_0^{N(t)} F(i,t)\, di.$$

We also define aggregate cumulative entry labor as

$$\bar{L}^{AE}(t) \equiv \int_0^{N(t)} L^{AE}(i,t)\, di. \tag{B.14}$$

Aggregating sector-level firm dynamics and accounting for the time variation in the frontier $N(t)$, the evolution of total firm mass satisfies

$$d\bar{F}(t) = \psi_e \bar{A}(t)\, d\bar{L}^{AE}(t) - \delta \bar{F}(t)\, dt.$$

On the other hand, by construction of the entry process, total firm mass also evolves according to

$$d\bar{F}(t) = \left(\int_0^{N(t)} e(i,t)\, di + \dot{N}(t) F_1 \right) dt - \delta \bar{F}(t)\, dt.$$

The first term inside the parentheses captures firm creation through entry within existing sectors, obtained by aggregating the Poisson arrival rates $e(i,t)$ across all interior sectors $i \leq N(t)$. The second term, $\dot{N}(t)F_1$, reflects instantaneous firm creation associated with frontier expansion: when the technological frontier advances at rate $\dot{N}(t)$, a mass F_1 of firms is created in each newly opened sector. Equating the two expressions for $d\bar{F}(t)$ yields

$$d\bar{L}^{AE}(t) = \frac{\int_0^{N(t)} e(i,t)\,di + \dot{N}(t)F_1}{\psi_e \bar{A}(t)}\,dt.$$

Since all terms on the right-hand side are locally bounded, $\bar{L}^{AE}(t)$ is absolutely continuous and therefore differentiable almost everywhere, with derivative given by

$$\dot{\bar{L}}^{AE}(t) = \frac{\int_0^{N(t)} e(i,t)\,di + \dot{N}(t)F_1}{\psi_e \bar{A}(t)}.$$

Finally, define aggregate entry labor at time t as $L^E(t) \equiv \dot{\bar{L}}^{AE}(t)$. Hence,

$$L^E(t) = \frac{\int_0^{N(t)} e(i,t)\,di + \dot{N}(t)F_1}{\psi_e \bar{A}(t)}. \tag{B.15}$$

B.3 Derivations and Proofs for Section 4.1

We first solve the consumer's problem and derive key properties of the function $\mathcal{L}$ in Appendix B.3.1, which summarizes the demand-side information. We then discuss the properties of the BGP in Appendix B.3.2. Finally, we detrend the entire dynamic system in Appendix B.3.3.

B.3.1 Properties of $\mathcal{L}$

We denote by $P_l(t)$ the price index of household l's final consumption. Consider the household's problem along a BGP with constant r:

$$\begin{aligned}
\max_{\{C_i(j,t)\}} \quad & \int_0^\infty e^{-\rho t}\frac{C_l(t)^{1-\sigma}}{1-\sigma}\,dt \\
\text{s.t.} \quad & C_l(t) = \int_0^{N(t)} i^{-\gamma} C_l(i,t)\,di + \min_{i\in(0,N_t]}\{C_l(i,t)\}\cdot X_l(t), \\
& P_l(t)C_l(t) = \int_0^{N(t)} P(i,t)C_l(i,t)\,di + P^X(t)X_l(t), \\
& \int_0^\infty e^{-rt}P_l(t)C_l(t)dt \le \int_0^\infty W(t)l\,e^{-rt}dt + \mathbb{V}(0)l.
\end{aligned}$$

The Euler equation is

$$r = \rho + \sigma\frac{\dot{C}_l}{C_l} + \frac{\dot{P}_l}{P_l}, \tag{B.16}$$

We make the following conjecture, which will be verified in the proof of Proposition 1 in Appendix B.4.

Conjecture 1 *For any household with endowment l, the equilibrium price index $P_l(t)$ is constant over time t.*

Suppose that Conjecture 1 holds. Then the Euler equation (B.16) simplifies to

$$r = \rho + \sigma \frac{\dot{C}_l}{C_l}, \tag{B.17}$$

Let μ_l denote the Lagrange multiplier associated with household l's budget constraint. The household's demand for good i is determined by the following reservation price condition:

$$C_l(i,t) = \begin{cases} 1 & \text{if } P(i,t) \le \kappa_l(i,t), \\ 0 & \text{if } P(i,t) > \kappa_l(i,t). \end{cases} \tag{B.18}$$

Solving the household's optimization problem, we have

$$\kappa_l(i,t) = \frac{i^{-\gamma} e^{-\rho t}}{\mu_l C_l(t)^{\sigma}} \cdot e^{rt}. \tag{B.19}$$

From equation (B.19), the household's willingness to pay $\kappa_l(i,t)$ is strictly decreasing in i. Therefore, if the product price $P(i,t)$ is strictly increasing in i, the household demand rule in equation (B.18) implies a cut-off structure: each household purchases all goods with index below some threshold and does not purchase goods with index above it.

At this stage, we also cannot establish the monotonicity of $P(i,t)$ from the demand side alone. We therefore proceed by conjecturing that the equilibrium price schedule is strictly increasing in i, and conduct the analysis under this conjectured property. We will later show that the equilibrium constructed from the supply side indeed satisfies this property, thereby validating the conjecture and closing all the arguments in this Section; see the end of Proposition 2 in Appendix B.5.

Conjecture 2 *At any time t, the equilibrium price $P(i,t)$ is strictly increasing in i.*

Lemma B.3 *Suppose that Conjecture 2 holds. At time t, for household with endowment l, there exists a positive number $\bar{i}(l,t)$, satisfying:*

$$C_l(i,t) = 1 \quad \textit{if and only if} \quad i \le \bar{i}(l,t),$$

Proof of Lemma B.3:

Fix a household with endowment l. Under Conjecture 2, the price schedule $P(i,t)$ is strictly increasing in i. From equation (B.19), the reservation price $\kappa_l(i,t)$ is strictly decreasing in i, with $\kappa_l(i,t) \to \infty$ as $i \to 0$. Hence, there are two cases.

Case 1: $\kappa_l(N(t),t) > P(N(t),t)$. Since $\kappa_l(i,t)$ is decreasing in i and $P(i,t)$ is increasing in i, for all $0 < i \le N(t)$ we have

$$\kappa_l(i,t) \ge \kappa_l(N(t),t) > P(N(t),t) \ge P(i,t).$$

Therefore, equation (B.18) implies that the household purchases all available goods. In this case, we set

$$\bar{i}(l,t) = N(t),$$

so that

$$C_l(i,t) = 1 \quad \text{if and only if} \quad i \leq \bar{i}(l,t).$$

Case 2: $\kappa_l(N(t),t) \leq P(N(t),t)$. Since $\kappa_l(i,t) \to \infty$ as $i \to 0$, while $P(i,t)$ is finite for any given $i > 0$, and since $\kappa_l(i,t)$ is strictly decreasing in i whereas $P(i,t)$ is strictly increasing in i, there exists a unique cutoff $\bar{i}(l,t) \in (0, N(t)]$ such that

$$\kappa_l(\bar{i}(l,t),t) = P(\bar{i}(l,t),t).$$

Moreover, for all $i < \bar{i}(l,t)$,

$$\kappa_l(i,t) > P(i,t),$$

and for all $i > \bar{i}(l,t)$,

$$\kappa_l(i,t) < P(i,t).$$

It then follows from equation (B.18) that

$$C_l(i,t) = 1 \quad \text{if and only if} \quad i \leq \bar{i}(l,t).$$

In either case, the household's consumption bundle is characterized by a cut-off rule. This completes the proof. ■

In what follows, it is convenient to characterize demand for good i by defining

$$\ell(i,t) = \inf\{l \in [\underline{l},\infty) : P(i,t) \leq \kappa_l(i,t)\} \tag{B.20}$$

to be the minimum labor endowment required for a household to purchase good i. By construction, when $l = \ell(i,t)$, the household is indifferent between purchasing and not purchasing good i, implying $P(i,t) = \kappa_l(i,t)$.

Lemma B.4 *Suppose Conjecture 2 holds. Then, for any* $\Delta t > 0$ *along the BGP, if two final goods* i_1 *and* i_2 *satisfy*

$$i_2 = e^{g_n \Delta t} i_1,$$

then

$$\ell(i_1,t) = \ell(i_2, t+\Delta t) \quad \text{and} \quad P(i_1,t) = e^{\gamma g_n \Delta t} P(i_2, t+\Delta t).$$

Proof of Lemma B.4:

First, we show that $\bar{i}(l,t)$ defined in Lemma B.3 must grows at the same rate as $N(t)$. Consider a household with endowment l at time t', who either have $X_l(t') = 0$ or $X_l(t') > 0$. Suppose $X_l(t') > 0$. Since $X_l(t)$ grows at a constant rate along the BGP, we have $X_l(t) > 0$ for all t. By equation (3), this household must have consumed all the available goods, i.e. $\bar{i}(l,t) = N(t)$, so $\bar{i}(l,t)$ grows at the rate g_n. In this case, the household's aggregate consumption is

$$C_l(t) = \int_0^{N(t)} i^{-\gamma} di + X_l(t) = \frac{N(t)^{1-\gamma}}{1-\gamma} + X_l(t).$$

Since $N(t)^{1-\gamma}$ grows at the rate $(1-\gamma)g_n$, to ensure $C_l(t)$ and $X_l(t)$ have constant growth rate, their growth rate must both be $(1-\gamma)g_n$.

Suppose $X_l(t') = 0$. Then we have $X_l(t) = 0$ for all t along the BGP. Thus, the household's aggregate consumption is given by

$$C_l(t) = \int_0^{\bar{i}(l,t)} i^{-\gamma} di = \frac{\bar{i}(l,t)^{1-\gamma}}{1-\gamma}.$$

Since $C_l(t)$ grows at a constant rate, $\bar{i}(l,t)$ must also grow at some constant rate. We claim that this rate is g_n. Given that the distribution of l is unbounded above, there must exist a positive measure of households that can afford all available goods and have $X_l(t) > 0$. As we have shown, the consumption of these households will grow at the rate $(1-\gamma)g_n$. By the definition of the aggregate final consumption $C(t)$ and the requirement that $C(t)$ grows at a constant rate on BGP, it must be that all households' consumption grows at the same constant rate $(1-\gamma)g_n$. This implies that $\bar{i}(l,t)$ grows at the rate g_n.

In either case above, $\bar{i}(l,t)$ grows at a constant rate g_n. Then, we show that $\ell(i_1,t) = \ell(i_2,t+\Delta t)$. Let $l = \ell(i_1,t)$, it follows that

$$\bar{i}(l,t+\Delta t) = e^{g_n \Delta t}\bar{i}(l,t) = e^{g_n \Delta t} i_1 = i_2.$$

By definition, $\ell(i_2,t+\Delta t)$ is the poorest household who is indifferent between consuming and not consuming good i_2 at time $t+\Delta t$. We claim that this household has endowment l. Suppose, for contradiction, that there exists $l' < l$ such that a household with endowment l' consumes good i_2 at time $t+\Delta t$. This implies $\bar{i}(l',t+\Delta t) \geq i_2$, and then $\bar{i}(l',t) \geq i_1$. Thus, at time t, the household with endowment $l' < l$ would consume good i_1, contradicting the definition of $l = \ell(i_1,t)$. Therefore,

$$\ell(i_1,t) = \ell(i_2,t+\Delta t).$$

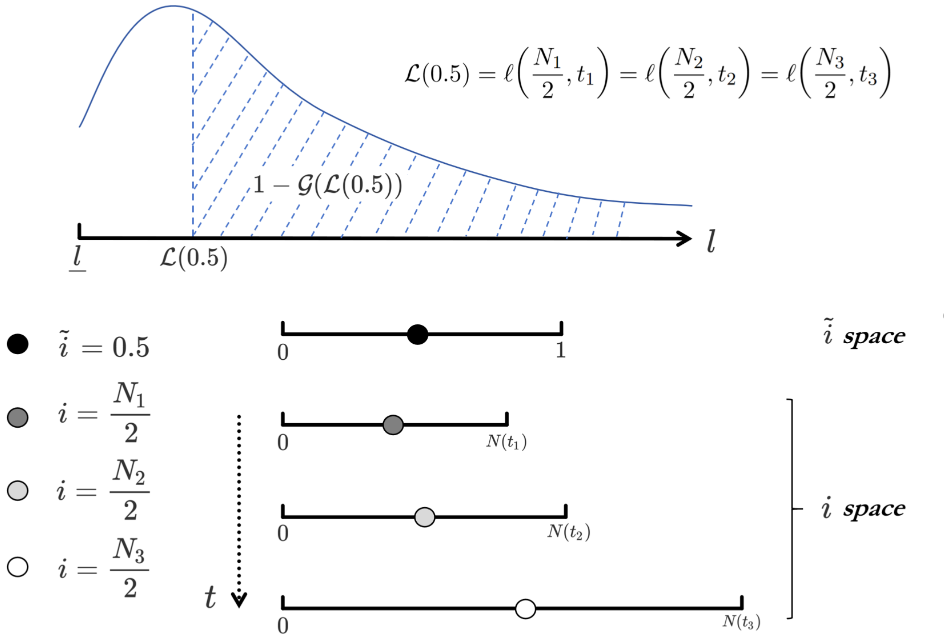
At last, we show that $P(i_1,t) = e^{\gamma g_n \Delta t} P(i_2,t+\Delta t)$. To see this, note that

$$\begin{aligned} P(i_2,t+\Delta t) &= \kappa_{\ell(i_2,t+\Delta t)}(i_2,t+\Delta t) \\ &= \frac{{i_2}^{-\gamma} e^{-\rho(t+\Delta t)}}{\mu_{\ell(i_2,t+\Delta t)} C_{\ell(i_2,t+\Delta t)}(t+\Delta t)^{\sigma}} \cdot e^{r(t+\Delta t)} \\ &= \frac{(e^{g_n \Delta t} i_1)^{-\gamma} e^{-\rho t}}{\mu_{\ell(i_1,t)} C_{\ell(i_1,t)}(t)^{\sigma}} \cdot e^{rt} \\ &= e^{-\gamma g_n \Delta t} P(i_1,t), \end{aligned}$$

which complete our proof. ■

By Lemma B.4, if $i_1/N(t_1) = i_2/N(t_2)$, then $\ell(i_1, t_1) = \ell(i_2, t_2)$. Thus, we can define functions $\mathcal{L}(\tilde{i}(t)) = \ell(i, t)$ and $p(\tilde{i}(t)) = e^{\gamma g_n t} P(i, t)$, where $\tilde{i}(t) = i/N(t)$. For simplicity, we write these function as $\mathcal{L}(\tilde{i})$ and $p(\tilde{i})$, and then present several useful lemmas on the properties of $\mathcal{L}$. First, Lemma B.5 determines the domain and codomain of $\mathcal{L}$, and Lemma B.6 shows the monotonicity of $\mathcal{L}$. Second, Lemma B.7 shows that, given $\mathcal{L}$, each household's consumption decision follows a simple threshold rule: if $l = \mathcal{L}(\tilde{i})$, the household consumes good j if and only if $j \le \tilde{i} N(t)$. Finally, Lemma B.8 characterizes the relationship between $\mathcal{L}$ and the equilibrium price $p(\tilde{i})$.

Figure B.2: Time Translation Invariance of ℓ



Notes. The upper panel plots the cumulative distribution of labor endowments across households. For a good with normalized index $\tilde{i} = 0.5$, the marginal household that purchases this good has a labor endowment level $\mathcal{L}(0.5)$. The lower panel provides a graphical representation of this translation: each horizontal line corresponds to the product space at a particular time t_j, ranging from 0 to $N(t_j)$. A fixed normalized index $\tilde{i} = 0.5$ (black dot) maps to different absolute product indices $i = N(t_j)/2$ (gray or white dots) over time.

Figure B.2 illustrates the *time translation invariance* on the demand side. The upper panel plots the cumulative distribution of labor endowments across households. For a good with normalized index $\tilde{i} = 0.5$, the marginal household that purchases this good has a labor endowment level $\mathcal{L}(0.5)$. The total demand for this good, $1 - \mathcal{G}(\mathcal{L}(0.5))$, is represented by the shaded area to the right of $\mathcal{L}(0.5)$ under the distribution curve. The equality

$$\mathcal{L}(0.5) = \ell\left(\frac{N_1}{2}, t_1\right) = \ell\left(\frac{N_2}{2}, t_2\right) = \ell\left(\frac{N_3}{2}, t_3\right)$$

shows that, along the BGP, the same normalized product index $\tilde{i} = 0.5$ corresponds to goods located at different absolute positions in the product space ($i = N_t/2$) at different times. As the number of varieties $N(t)$ expands over time, moving forward in time by Δt is equivalent to shifting along the product space by a factor of $e^{g_n \Delta t}$.

The lower panel provides a graphical representation of this translation: each horizontal line corresponds to the product space at a particular time t_j, ranging from 0 to $N(t_j)$. A fixed normalized index $\tilde{i} = 0.5$ (black dot) maps to different absolute product indices $i = N(t_j)/2$ (gray or white dots) over time. Thus, the figure demonstrates that the demand for a given normalized product type remains invariant over time once expressed in terms of $\tilde{i}$ rather than the absolute index i.

Lemma B.5 *Suppose Conjecture 2 holds, there exist constants $\eta > 0$ and $\bar{l} < \infty$ such that:*

$$\mathcal{L}(\tilde{i}) = \underline{l} \quad \textit{if and only if} \quad 0 < \tilde{i} \leq \eta,$$

and

$$\mathcal{L}(1) = \bar{l}.$$

Proof of Lemma B.5:

We first establish the existence of $\bar{l}$. To do so, note that the set of households who purchase goods $N(t)$ is nonempty. This follows from the fact that endowment l is unbounded, whereas $N(t)$ is finite. Hence, by the household budget constraint, there exists a finite endowment level above which households purchase all goods, including goods $N(t)$. We therefore can define

$$\bar{l} \equiv \inf \left\{ l : C_l(N(t), t) = 1 \right\}.$$

By the definition of $\mathcal{L}(\cdot)$, it follows that

$$\mathcal{L}(1) = \bar{l}.$$

Next, we demonstrate the existence of η. Define $\underline{\kappa}(i,t) = \inf_l \kappa_l(i,t)$, so that product i is consumed by all households if and only if $P(i,t) \leq \underline{\kappa}(i,t)$, as equation (B.18) implies. Under Conjecture 2, and $\underline{\kappa}(i,t)$ is decreasing in i with $\lim_{i \to 0} \underline{\kappa}(i,t) = \infty$, there exists a unique value $i = \tilde{\eta}(t)$ at each time such that

$$P(i,t) = \underline{\kappa}(i,t)$$

and

$$P(i,t) \leq \underline{\kappa}(i,t) \quad \text{if and only if} \quad 0 < i \leq \tilde{\eta}(t).$$

Thus, we have

$$\mathcal{L}\left(\frac{i}{N(t)}\right) = \underline{l} \quad \text{if and only if} \quad 0 < i \leq \tilde{\eta}(t).$$

As we have shown in Lemma B.4, $\mathcal{L}\left(\frac{i}{N(t)}\right)$ should be independent of time t, which implies $\tilde{\eta}(t)$ must grow at rate g_n. Therefore, we define $\eta = \frac{\tilde{\eta}(t)}{N(t)}$, which implies:

$$\mathcal{L}(\tilde{i}) = \underline{l} \quad \text{if and only if} \quad 0 < \tilde{i} \leq \eta.$$

This completes the proof. ■

In Lemma B.5, we establish the existence of a constant fraction η such that, for all $i \leq \eta N(t)$, every household purchases good i. Hence, subsequent analysis can primarily focus on the domain $\tilde{i} \in [\eta, 1]$.

Lemma B.6 *Suppose that Conjecture 2 holds, function* $\mathcal{L}(\tilde{i})$ *is strictly increasing on* $[\eta, 1]$.

Proof of Lemma B.6:

Take any $\eta \leq \tilde{i}_1 < \tilde{i}_2 \leq 1$. Suppose for contradiction, that $\mathcal{L}(\tilde{i}_1) \geq \mathcal{L}(\tilde{i}_2)$. By definition of $\mathcal{L}$, this implies

$$\ell_1 \equiv \ell(\tilde{i}_1 N(t), t) \geq \ell(\tilde{i}_2 N(t), t) \equiv \ell_2.$$

By construction, a household with endowment ℓ_1 is indifferent at good $\tilde{i}_1 N(t)$, i.e.,$P(\tilde{i}_1 N(t), t) = \kappa_{\ell_1}(\tilde{i}_1 N(t), t)$. Under Conjecture 2, this household will only consume goods with $0 < i \leq \tilde{i}_1 N(t)$. Similarly, household with endowment ℓ_2 will only consume goods with $0 < i \leq \tilde{i}_2 N(t)$. Because $\tilde{i}_1 < \tilde{i}_2$, the consumption set of household ℓ_1 is a strict subset of that of household ℓ_2. Hence, the bundle chosen by household ℓ_2 is strictly preferred by household ℓ_1 under the same preference. Moreover, since $\ell_1 \geq \ell_2$, household ℓ_1 can afford the bundle chosen by household ℓ_2 whenever household ℓ_2 can. Therefore, the bundle associated with cutoff $\tilde{i}_1 N(t)$ cannot be optimal for household ℓ_1, a contradiction. Therefore, $\mathcal{L}(\tilde{i})$ must be strictly increasing on $[\eta, 1]$. ■

A direct implication of Lemma B.6 is that, for any given i, the willingness to pay $\kappa_l(i, t)$ is strictly increasing in l over $[\underline{l}, \bar{l}]$. Moreover, Lemma B.6 implies that $\mathcal{L}(\tilde{i})$ is invertible on $[\eta, 1]$. We therefore define its inverse function $\mathcal{L}^{-1} : [\underline{l}, \bar{l}] \to [\eta, 1]$ by

$$\mathcal{L}^{-1}(l) = \tilde{i} \quad \text{if and only if} \quad \mathcal{L}(\tilde{i}) = l. \tag{B.21}$$

In the next lemma, we show that the household consumption bundle is characterized by $\mathcal{L}^{-1}(l)$.

Lemma B.7 *Suppose that Conjecture 1 and Conjecture 2 hold. Then, for a household with endowment* $\underline{l} < l < \bar{l}$, *its optimal consumption rule is given by:*

$$C_l(i, t) = \begin{cases} 1 & \textit{if } i \leq \mathcal{L}^{-1}(l) N(t), \\ 0 & \textit{if } i > \mathcal{L}^{-1}(l) N(t), \end{cases}$$

and for the luxury consumption:

$$X_l(t) = 0.$$

For a household with endowment $l \geq \bar{l}$, *its optimal consumption rule is given by*

$$C_l(i, t) = 1 \quad \textit{for all } i \leq N(t),$$

and its luxury service is given by

$$X_l(t) = \frac{Y(t)(l - \bar{l})}{P^X(t)}. \tag{B.22}$$

Proof of Lemma B.7:

Step 1. For a household with endowment $\underline{l} < l < \bar{l}$, the definition of $\mathcal{L}^{-1}(l)$ implies that:

$$P(\mathcal{L}^{-1}(l)N(t), t) = \kappa_l(\mathcal{L}^{-1}(l)N(t), t).$$

Under Conjecture 2, and the fact that $\kappa_l(i,t)$ is strictly decreasing in i, it follows that:

$$P(i,t) < \kappa_l(i,t) \quad \text{for all } i < \mathcal{L}^{-1}(l)N(t),$$

$$P(i,t) > \kappa_l(i,t) \quad \text{for all } i > \mathcal{L}^{-1}(l)N(t).$$

By equation (B.18), the household therefore consumes all goods with $i \leq \mathcal{L}^{-1}(l)N(t)$ and consumes no goods with $i > \mathcal{L}^{-1}(l)N(t)$. Finally, it is straightforward to see that $X_l(t) = 0$, since household l consumes no goods with $i > \mathcal{L}^{-1}(l)N(t)$, we have

$$\min_{i \in (0, N(t)]} C_l(i,t) = 0.$$

That is, luxury service consumption yields no utility at the margin.

Step 2. For a household with endowment $l \geq \bar{l}$, Lemma B.5 implies that the household consumes the good indexed by $N(t)$. By the same logic as in **Step 1**, it then follows that the household must consume all goods with $i \leq N(t)$.

To derive their luxury service consumption, we first rewrite the household budget constraint in per-period form. The present-value budget constraint for a household with endowment l is

$$\int_0^\infty e^{-rt} P_l C_l(t) dt = \int_0^\infty W(t) l \cdot e^{-rt} dt + \mathbb{V}(0) l.$$

Under Conjecture 1, P_l is the (constant) price index of household l's final consumption. Since $\mathbb{V}(0)$ denotes the aggregate market value of all firms at $t = 0$, we have

$$\mathbb{V}(0) \equiv \int_0^\infty e^{-rt} \left\{ \int_0^{N(t)} \left[\Pi(i,t) + \int_0^{F(i,t)} \Pi_i(j,t) dj \right] di - W(t) L^{RD}(t) \right\} dt, \tag{B.23}$$

where the total labor employed in R&D is given by

$$L^{RD}(t) = \int_0^{N(t)} \int_0^{F(i,t)} L_i^I(j,t) dj \, di + L^E(t) + L^N(t).$$

Substituting $\mathbb{V}(0)$ into the present-value constraint yields

$$\int_0^\infty e^{-rt} P_l C_l(t) dt = \int_0^\infty e^{-rt} \left\{ W(t) + \int_0^{N(t)} \left[\Pi(i,t) + \int_0^{F(i,t)} \Pi_i(j,t) dj \right] di - W(t) L^{RD}(t) \right\} l dt.$$

Since labor is the only input in the economy, the resource constraint implies that the total net profit flow across all firms must be equal to aggregate output minus labor income

$$\int_0^{N(t)} \left[\Pi(i,t) + \int_0^{F(i,t)} \Pi_i(j,t) dj \right] di - W(t) L^{RD}(t) = Y(t) - W(t).$$

So the present-value constraint simplifies to

$$\int_0^\infty e^{-rt} P_l C_l(t)\, dt = \int_0^\infty e^{-rt} Y(t)\, l\, dt. \tag{B.24}$$

Let $a_l(t)$ denote the household's net saving with $a_l(0) = 0$. The corresponding flow budget constraint is

$$\dot{a}_l(t) + P_l C_l(t) = r a_l(t) + Y(t) l. \tag{B.25}$$

We now show that $a_l(t) \equiv 0$ for all t and all l along the BGP. Under Conjecture 1, the Euler equation (B.17) implies all households' consumption grows at the common rate

$$g_c = \frac{\dot{C}_l(t)}{C_l(t)} = \frac{r - \rho}{\sigma}.$$

Along the BGP, aggregate income grows at rate g_y, i.e. $Y(t) = Y(0) e^{g_y t}$, and consumption expenditure satisfies $P_l C_l(t) = P_l C_l(0) e^{g_c t}$. Substituting these paths into (B.24) yields

$$\frac{P_l C_l(0)}{r - g_c} = \frac{Y(0) l}{r - g_y}. \tag{B.26}$$

Suppose $g_c > g_y$. Then (B.26) implies $P_l C_l(0) < Y(0) l$ for all l, and evaluating (B.25) at $t = 0$ gives

$$\dot{a}_l(0) > 0 \qquad \text{for all } l.$$

Therefore $\int \dot{a}_l(0)\, d\mathcal{G}(l) > 0$, which contradicts the credit-market clearing condition that the traded asset is in zero net supply:

$$\int a_l(t)\, d\mathcal{G}(l) = 0 \quad \text{for all } t.$$

An analogous contradiction arises if $g_c < g_y$. Hence we must have $g_c = g_y$. Then (B.26) implies $P_l C_l(0) = Y(0) l$, and since both sides grow at the same rate, it follows that

$$P_l C_l(t) = Y(t) l \qquad \text{for all } t \text{ and } l. \tag{B.27}$$

Substituting (B.27) into (B.25) yields $\dot{a}_l(t) = r a_l(t)$. Together with $a_l(0) = 0$, we obtain $a_l(t) \equiv 0$

for all t and l, and the per-period budget constraint simplifies to

$$\int_0^{N(t)} P(i,t)C_l(i,t)\,di + P^X(t)X_l(t) = Y(t)l. \tag{B.28}$$

From equation (B.27), for a household with endowment $l \geq \bar{l}$, we have

$$\int_0^{N(t)} P(i,t)\,di + P^X(t)X_l(t) = Y(t)l.$$

and for the household with endowment $\bar{l}$, we have

$$\int_0^{N(t)} P(i,t)\,di = Y(t)\bar{l}.$$

Combining the above two equations yields equation (B.22). ■

Lemma B.8 *Suppose that Conjecture 1 and Conjecture 2 hold. Then the equilibrium price is related to the marginal value of $\mathcal{L}$:*

$$p(\tilde{i}) = y\mathcal{L}'(\tilde{i}) \quad \textit{for} \quad \tilde{i} \in [\eta, 1],$$

where the detrended price is defined as $p(\tilde{i}) = e^{\gamma g_n t}P(i,t)$.

Proof of Lemma B.8:

We solve the detrended price level $p(\tilde{i})$ using equation (B.28). Under Conjecture 1, combing equation (B.19) with the Euler equation (B.17) yields

$$\kappa_l(i,t) = \frac{i^{-\gamma}}{\tilde{\mu}_l}, \quad \text{with} \quad \tilde{\mu}_l = \mu_l C_l(0)^{\sigma}.$$

which implies $\kappa_l(i,t)$ is independent of time t. For $\tilde{i} \geq \eta$, we have

$$P(i,t) = \frac{i^{-\gamma}}{\tilde{\mu}_l} \quad \text{for} \quad l = \ell(i,t).$$

A household with endowment $\underline{l} \leq l \leq \mathcal{L}(1)$ consumes up to $\mathcal{L}^{-1}(l)N(t)$ and has zero luxury service $X_l(t) = 0$, where $\mathcal{L}^{-1}(l)$ is defined in equation (B.21). So equation (B.28) simplifies to

$$\int_0^{\mathcal{L}^{-1}(l)N(t)} P(i,t)di = Y(t)l \quad \text{for} \quad l \in [\underline{l}, \mathcal{L}(1)],$$

and detrended yields

$$\int_0^{\mathcal{L}^{-1}(l)} \frac{\tilde{i}^{-\gamma}}{\tilde{\mu}_{\mathcal{L}(\tilde{i})}} d\tilde{i} = yl \quad \text{for} \quad l \in [\underline{l}, \mathcal{L}(1)].$$

Differentiating with respect to l gives

$$\tilde{\mu}_l = \frac{\left[\mathcal{L}^{-1}(l)\right]^{-\gamma}}{y\,\mathcal{L}'(\mathcal{L}^{-1}(l))} \quad \text{for} \quad l \in [\underline{l}, \mathcal{L}(1)],$$

or equivalently,

$$\tilde{\mu}_{\mathcal{L}(\tilde{i})} = \frac{\tilde{i}^{-\gamma}}{y\mathcal{L}'(\tilde{i})} \quad \text{for} \quad \tilde{i} \in [\eta, 1].$$

Using the detrended expression of $p(\tilde{i}) = \frac{\tilde{i}^{-\gamma}}{\tilde{\mu}_{\mathcal{L}(\tilde{i})}}$, we obtain

$$p(\tilde{i}) = y\mathcal{L}'(\tilde{i}) \quad \text{for} \quad \tilde{i} \in [\eta, 1].$$

■

B.3.2 Properties of the BGE

In this appendix, we derive properties of the BGE. Let $\{g_c, g_x, g_y, g_w, g_n\}$ denote the growth rates of $\{C, X, Y, w, N\}$. The next lemma pins down their relationship on the BGP.

Lemma B.9 *Along the BGP, all aggregate quantities grow at constant rates. In particular,*

$$g_c = g_x = g_y = g_w = (1-\gamma)g_n.$$

Moreover, the luxury service price P^X satisfies

$$P^X(t) = \frac{W(t)}{\bar{A}^{1-\gamma}}, \tag{B.29}$$

which is constant along the BGP.

Proof of Lemma B.9:

We have shown that the growth rate of $C_l(t)$ is $(1-\gamma)g_n$ in Lemma B.4, and also equal to g_y in Lemma B.7. By the definition of $C(t)$, the growth rate of $C_l(t)$ is equal to the growth rate of $C(t)$. Therefore, we have established that $g_c = g_y = (1-\gamma)g_n$.

Next we show that $g_y = g_w$. It suffices to show that the aggregate labor share is constant. In the luxury service sector, the resource constraint implies that all output is distributed to labor, yielding a constant labor share of 1. In the production sector, the aggregate output is divided into three components: (i) profits from final-goods production, (ii) profits from intermediate-goods production, and (iii) labor compensation. As shown in equation (B.3), the labor share in the production sector across all final goods is constant and equal to $(1-\beta)^2$. Furthermore, Lemma B.7 implies the relative size of the luxury service sector is constant along the stationary equilibrium, so the aggregate labor share is time-invariant. With fixed labor supply, it follows that $g_y = g_w$.

Finally, the first-order condition for the luxury service producer problem gives equation (B.29). Because $g_w = (1-\gamma)g_n$, $P^X(t)$ is constant along the BGP. As shown in Lemma B.7, each household with $l < \bar{l}$ has $X_l(t) = 0$, and each household with $l \geq \bar{l}$ has $X_l(t) > 0$, which grows at rate $(1-\gamma)g_n$ by equation (B.22). The market-clearing condition for luxury service implies that the growth rate of $X(t)$ is $(1-\gamma)g_n$. ■

To further simplify the analysis, it is convenient to exploit the symmetry property of the BGP, which allows the product space to be normalized from $(0, N(t)]$ to $(0, 1]$. In the following lemma, we show an important property of our BGP system: time-translation invariance, which implies a simple invariance: moving forward in time by Δt is equivalent to moving along the product ladder by the factor $e^{g_n \Delta t}$.

Lemma B.10 *Along the BGP, for any $\Delta t > 0$, if two categories of final goods i_1 and i_2 satisfies*

$$i_2 = e^{g_n \Delta t} i_1,$$

then the following relationships hold:

$$\begin{aligned} Y(i_1, t) &= Y(i_2, t + \Delta t), & z_{i_1}(t) &= z_{i_2}(t + \Delta t), \\ F(i_1, t) &= F(i_2, t + \Delta t), & B(i_1, t) &= e^{-\frac{1-\beta}{\beta} g_n \Delta t} B(i_2, t + \Delta t). \end{aligned}$$

Proof of Lemma B.10:

We will complete the proof in two steps. First, we show that the firm value V satisfies a time translation invariance condition,

$$V_{i_2}(t + \Delta t) = e^{-\left(\gamma + \theta \frac{1-\beta}{\beta}\right) g_n \Delta t} V_{i_1}(t).$$

Second, we show that z, Y, B, and F also satisfy time translation invariance conditions.

Step 1 Show the translation rule for V.

We first establish a time translation invariance property for firm value V. We need to derive the growth rate of sectoral productivity $B(i, t)$ in equation (19). Let

$$X(i, t) \equiv \int_0^{F(i,t)} \tilde{A}_i(j, t)^{\theta \frac{1-\beta}{\beta}} \, dj.$$

Considering a small time interval dt, we partition firms in sector i into two groups: incumbent firms indexed by $j \in [0, F(i, t)]$, and newly entering firms indexed by $j \in (F(i, t), F(i, t + dt)]$. For an incumbent firm $j \in [0, F(i, t)]$, productivity evolves according to an independent Poisson process $N_i^z(j, t)$, which denotes successful innovation. For an incumbent firm that remains active, we intuitively have

$$d\left(\tilde{A}_i(j, t)^{\theta \frac{1-\beta}{\beta}}\right) = \left(\lambda^{\theta \frac{1-\beta}{\beta}} - 1\right) \tilde{A}_i(j, t)^{\theta \frac{1-\beta}{\beta}} \, dN_i^z(j, t).$$

Taking conditional expectations, we obtain

$$\mathbb{E}_t\Big[d\left(\tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}}\right)\Big] = z_i(t)\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right)\tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}}\,dt.$$

Given that incumbent firms are subject to independent idiosyncratic shocks, by an exact law of large numbers, the aggregate realized flow equals its conditional expectation almost surely

$$\begin{aligned}\int_0^{F(i,t)} d\Big(\tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}}\Big)\,dj &= \int_0^{F(i,t)} \mathbb{E}_t\Big[d\Big(\tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}}\Big)\Big]\,dj. \\ &= z_i(t)\big(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\big)X(i,t)\,dt.\end{aligned}$$

For newly entering firms, we assume that entrants draw their productivity from the current distribution of incumbents. Therefore, for any $j \in (F(i,t), F(i,t+dt)]$, we have

$$\mathbb{E}_t\Big[\tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}}\Big] = \frac{1}{F(i,t)}\int_0^{F(i,t)} \tilde{A}_i(j',t)^{\theta\frac{1-\beta}{\beta}}\,dj' = \frac{X(i,t)}{F(i,t)}.$$

Since the measure of entrants over dt is $e(i,t)\,dt$, the total contribution of entry is

$$e(i,t)\,dt \cdot \frac{X(i,t)}{F(i,t)}.$$

Since exit shocks are independent of firm productivity, exiting firms constitute a random sample of incumbents. Hence, by the law of large numbers, the expected productivity mass lost to exit is

$$\delta F(i,t)\cdot\frac{X(i,t)}{F(i,t)}dt.$$

By Ito's Lemma, we obtain

$$\begin{aligned}dX(i,t) &= dF(i,t)\cdot\frac{X(i,t)}{F(i,t)} + \int_0^{F(i,t)} d\Big(\tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}}\Big)\,dj \\ &= (e(i,t) - \delta F(i,t))\frac{X(i,t)}{F(i,t)}dt + z_i(t)\big(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\big)X(i,t)dt.\end{aligned} \tag{B.30}$$

Dividing both sides by $X(i,t)$, and substituting into

$$B(i,t) = \bar{A}(t)^{(1-\theta)\frac{1-\beta}{\beta}}X(i,t),$$

we obtain equation (19).

The entry condition for intermediate firms implies

$$W(t) = \bar{A}(t)\psi_e\left[\frac{\int_0^{F(i,t)}\tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}}dj}{F(i,t)}\right]V_i(t)$$

$$= \quad \bar{A}(t)^{1-(1-\theta)\frac{1-\beta}{\beta}} \, \psi_e \, \frac{B(i,t)}{F(i,t)} V_i(t),$$

where it follows from a law of large numbers that

$$\frac{\int_0^{F(i,t)} \tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}} dj}{F(i,t)} = \mathbb{E}[\tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}}]$$

represents the expected productivity drawn by an entrant from the distribution of incumbent firms' productivities in sector i. By Ito's Lemma,

$$\frac{\dot{W}(t)}{W(t)} = \left(1-(1-\theta)\frac{1-\beta}{\beta}\right)\frac{\dot{\bar{A}}(t)}{\bar{A}(t)} + \frac{\dot{B}(i,t)}{B(i,t)} - \frac{\dot{F}(i,t)}{F(i,t)} + \frac{\dot{V}_i(t)}{V_i(t)}. \tag{B.31}$$

Combining with equation (19) and (B.31), we obtain

$$\frac{\dot{V}_i(t)}{V_i(t)} = \frac{\dot{W}(t)}{W(t)} - \frac{\dot{\bar{A}}(t)}{\bar{A}(t)} - z_i(j,t)\cdot\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right), \tag{B.32}$$

where $z_i(j,t)$ is given by equation (B.5).

From the firm's HJB equation in the proof of Lemma B.1, $V_i(t)$ satisfies

$$\begin{aligned} rV_i(t) - \dot{V}_i(t) &= \beta(1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{P(i,t)^{\frac{1}{1-\beta}}}{W(t)}\bar{A}(t)^{1-\theta}\right]^{\frac{1-\beta}{\beta}} \\ &\quad - W(t)\cdot\left[\frac{W(t)}{(\lambda^{\theta\frac{1-\beta}{\beta}}-1)V_i(t)\psi\alpha\bar{A}(t)^{\xi}}\right]^{\frac{1}{\alpha-1}} + z_i(j,t)(\lambda^{\theta\frac{1-\beta}{\beta}}-1)V_i(t) - \delta V_i(t), \end{aligned}$$

where ξ is a parameter defined in equation (B.8). Substituting the expression for $\dot{V}_i(t)$ in equation (B.32) and simplifying, we find

$$\begin{aligned} \left[\rho+\delta+(\sigma(1-\gamma)+\gamma)g_n\right]V_i(t) &= \beta(1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{P(i,t)^{\frac{1}{1-\beta}}}{W(t)}\bar{A}(t)^{1-\theta}\right]^{\frac{1-\beta}{\beta}} \\ &\quad - W(t)\cdot\left[\frac{W(t)}{(\lambda^{\theta\frac{1-\beta}{\beta}}-1)V_i(t)\psi\alpha\bar{A}(t)^{\xi}}\right]^{\frac{1}{\alpha-1}}, \end{aligned} \tag{B.33}$$

where we have used Lemma B.9 and the Euler equation $r = \rho + \sigma(1-\gamma)g_n$ along the BGP.

Next, we will show that V satisfies a time translation invariance condition based on equation (B.33). Consider the index translation $i_2 = e^{g_n \Delta t} i_1$. For ease of notation, we ignore the constant multiplier $[\rho+\delta+(\sigma(1-\gamma)+\gamma)g_n]$. Then $V_{i_1}(t)$ and $V_{i_2}(t)$ are given by

$$V_{i_1}(t) = \beta(1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{P(i_1,t)^{\frac{1}{1-\beta}}}{W(t)}\bar{A}(t)^{(1-\theta)}\right]^{\frac{1-\beta}{\beta}}$$

$$
\begin{aligned}
&\qquad -W(t)\cdot\left[\frac{W(t)}{\left(\lambda^{\theta\frac{1-\beta}{\beta}}-1\right)V_{i_1}(t)\cdot\psi\alpha\bar{A}(t)^{\xi}}\right]^{\frac{1}{\alpha-1}},\\
V_{i_2}(t+\Delta t) &= \beta(1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{P(i_2,t+\Delta t)^{\frac{1}{1-\beta}}}{W(t+\Delta t)}\bar{A}(t+\Delta t)^{(1-\theta)}\right]^{\frac{1-\beta}{\beta}}\\
&\qquad -W(t+\Delta t)\cdot\left[\frac{W(t+\Delta t)}{\left(\lambda^{\theta\frac{1-\beta}{\beta}}-1\right)V_{i_2}(t+\Delta t)\cdot\psi\alpha\bar{A}(t+\Delta t)^{\xi}}\right]^{\frac{1}{\alpha-1}}.
\end{aligned}
$$

To show the relationship between $V_{i_1}(t)$ and $V_{i_2}(t+\Delta t)$, observe that Lemma B.4 implies $P(i_2,t+\Delta t)=e^{-\gamma g_n\Delta t}P(i_1,t)$, and the aggregate variable evolves according to $W(t+\Delta t)=e^{(1-\gamma)g_n\Delta t}W(t)$, $\bar{A}(t+\Delta t)=e^{g_n\Delta t}\bar{A}(t)$. Substituting these expressions into $V_{i_2}(t)$, we have

$$
\begin{aligned}
V_{i_2}(t+\Delta t) &= \beta(1-\beta)^{\frac{1-\beta}{\beta}}e^{-\left(\gamma+\theta\frac{1-\beta}{\beta}\right)g_n\Delta t}\left[\frac{P(i_1,t)^{\frac{1}{1-\beta}}}{W(t)}\bar{A}(t)^{(1-\theta)}\right]^{\frac{1-\beta}{\beta}}\\
&\qquad -e^{(1-\gamma)g_n\Delta t}W(t)\cdot\left[\frac{e^{(1-\gamma)g_n\Delta t}W(t)}{\left(\lambda^{\theta\frac{1-\beta}{\beta}}-1\right)V_{i_2}(t+\Delta t)\cdot\psi\alpha e^{\xi g_n\Delta t}\bar{A}(t)^{\xi}}\right]^{\frac{1}{\alpha-1}}.
\end{aligned}
$$

Multiply each side by $e^{\left(\gamma+\theta\frac{1-\beta}{\beta}\right)g_n\Delta t}$ to derive

$$
\begin{aligned}
e^{\left(\gamma+\theta\frac{1-\beta}{\beta}\right)g_n\Delta t}V_{i_2}(t+\Delta t) &= \beta(1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{P(i_1,t)^{\frac{1}{1-\beta}}}{W(t)}\bar{A}(t)^{(1-\theta)}\right]^{\frac{1-\beta}{\beta}}\\
&\qquad -W(t)\cdot\left[\frac{e^{\left[(1-\gamma)+(\alpha-1)\left(\theta\frac{1-\beta}{\beta}+1\right)-\xi\right]g_n\Delta t}W(t)}{\left(\lambda^{\theta\frac{1-\beta}{\beta}}-1\right)V_{i_2}(t+\Delta t)\cdot\psi\alpha\bar{A}(t)^{\xi}}\right]^{\frac{1}{\alpha-1}}\\
&= \beta(1-\beta)^{\frac{1-\beta}{\beta}}\left[\frac{P(i_1,t)^{\frac{1}{1-\beta}}}{W(t)}\bar{A}(t)^{(1-\theta)}\right]^{\frac{1-\beta}{\beta}}\\
&\qquad -W(t)\cdot\left[\frac{W(t)}{\left(\lambda^{\theta\frac{1-\beta}{\beta}}-1\right)e^{\left(\gamma+\theta\frac{1-\beta}{\beta}\right)g_n\Delta t}V_{i_2}(t+\Delta t)\cdot\psi\alpha\bar{A}(t)^{\xi}}\right]^{\frac{1}{\alpha-1}}.
\end{aligned}
$$

Thus, We have

$$V_{i_2}(t+\Delta t)=e^{-\left(\gamma+\theta\frac{1-\beta}{\beta}\right)g_n\Delta t}V_{i_1}(t).$$

This completes the first step of the proof.

Step 2 Show temporal translation for $z_i(j,t),Y(i,t),B(i,t),F(i,t)$.

1. Innovation intensity $z_i(j,t)$. Using the definition of $z_i(j,t)$ and the temporal translation of W, V, and $\bar{A}$, we have

$$\begin{aligned} z_{i_2}(j, t+\Delta t) &= \psi \left[\frac{W(t+\Delta t)}{(\lambda^{\theta \frac{1-\beta}{\beta}} - 1) V_{i_2}(t+\Delta t) \cdot \psi \alpha \bar{A}(t+\Delta t)^{\frac{\xi}{\alpha}}} \right]^{\frac{\alpha}{\alpha-1}} \\ &= \psi \left[\frac{e^{(1-\gamma) g_n \Delta t} W(t)}{(\lambda^{\theta \frac{1-\beta}{\beta}} - 1) e^{-(\gamma + \theta \frac{1-\beta}{\beta}) g_n \Delta t} V_{i_1}(t) \cdot \psi \alpha e^{\frac{\xi}{\alpha} g_n \Delta t} \bar{A}(t)^{\frac{\xi}{\alpha}}} \right]^{\frac{\alpha}{\alpha-1}} \\ &= \psi \left[\frac{W(t)}{(\lambda^{\theta \frac{1-\beta}{\beta}} - 1) V_{i_1}(t) \cdot \psi \alpha \bar{A}(t)^{\frac{\xi}{\alpha}}} \right]^{\frac{\alpha}{\alpha-1}} = z_{i_1}(j, t). \end{aligned}$$

2. **Sectoral output** $Y(i, t)$**.** Using $Y(i, t) = 1 - \mathcal{G}(\mathcal{L}(\tilde{i}))$ and the fact that $\tilde{i}_1 = \tilde{i}_2$, we get

$$Y(i_2, t+\Delta t) = 1 - \mathcal{G}(\mathcal{L}(\tilde{i}_2)) = 1 - \mathcal{G}(\mathcal{L}(\tilde{i}_1)) = Y(i_1, t).$$

3. **Sectoral productivity** $B(i, t)$**.** By definition,

$$\begin{aligned} B(i_2, t+\Delta t) &= \frac{Y(i_2, t+\Delta t) \left[\frac{W(t+\Delta t)}{P(i_2, t+\Delta t)} \right]^{\frac{1-\beta}{\beta}}}{(1-\beta)^{\frac{1-2\beta}{\beta}}} = \frac{Y(i_1, t) \left[\frac{e^{g_n \Delta t} W(t)}{P(i_1, t)} \right]^{\frac{1-\beta}{\beta}}}{(1-\beta)^{\frac{1-2\beta}{\beta}}} \\ &= e^{\frac{1-\beta}{\beta} g_n \Delta t} B(i_1, t). \end{aligned}$$

4. **Firm number** $F(i, t)$**.** Using the entry condition,

$$\begin{aligned} F(i_2, t+\Delta t) &= \bar{A}(t+\Delta t)^{1-(1-\theta)\frac{1-\beta}{\beta}} \psi_e \frac{B(i_2, t+\Delta t)}{W(t+\Delta t)} V_{i_2}(t+\Delta t) \\ &= e^{[1-(1-\theta)\frac{1-\beta}{\beta}] g_n \Delta t} \bar{A}(t)^{1-(1-\theta)\frac{1-\beta}{\beta}} \psi_e \frac{e^{\frac{1-\beta}{\beta} g_n \Delta t} B(i_1, t)}{e^{(1-\gamma) g_n \Delta t} W(t)} e^{-(\gamma + \theta \frac{1-\beta}{\beta}) g_n \Delta t} V_{i_1}(t) \\ &= F(i_1, t). \end{aligned}$$

Hence, we have verified that $z_i(j, t)$, $Y(i, t)$, $F(i, t)$, and $B(i, t)$ all satisfy the time translation invariance conditions. ■

To better illustrate this lemma, define the normalized product index $\tilde{i}(t) \equiv \frac{i}{N(t)}$, which measures the relative luxury level of sector i at time t. Lemma B.10 implies that two sectors with the same $\tilde{i}$ at different points in time will share identical (possibly detrended) values for output, innovation rates, firm numbers, and other sector-specific variables. This property can be interpreted as a *time translation invariance*: shifting forward in time by Δt is equivalent to moving along the product space by a factor of $e^{g_n \Delta t}$. Based on that, we can translate all the dynamics of a specific sector into a set of differential equations across sectors. Building on this property, we proceed by detrending all sector-level variables by their respective asymptotic growth rates in the next subsection.

B.3.3 Full Detrending Procedure

To characterize the BGE, we eliminate the common exponential growth components of all variables and transform the system into a stationary representation. The detrending procedure proceeds in three steps.

Step 1. Identifying asymptotic growth rates. We first compute the asymptotic growth rates of all aggregate- and sector-level variables. For example, the measure of varieties $N(t)$ grows at rate g_n, sectoral productivity $B(i,t)$ grows at rate $\frac{1-\beta}{\beta}g_n$, and wages $W(t)$ grow at rate $(1-\gamma)g_n$. From these primitives, we can infer the growth trend of any endogenous variable in the system.

Step 2. Applying time translation invariance. Lemma B.10 shows that along the BGP, shifting the good index by a factor of $e^{g_n\Delta t}$ while moving time forward by Δt preserves the structure of the economy. This invariance ensures that the system admits a representation in which all real allocations are invariant under such translations and differ only by deterministic exponential factors.

Step 3. Constructing detrended variables. Using the identified growth rates and translation invariance, we define stationary counterparts of all variables by removing their exponential trends.

For aggregate variables, we set detrended variables as

$$y = e^{-(1-\gamma)g_n t}Y(t), \quad w = e^{-(1-\gamma)g_n t}W(t), \quad 1 = e^{-g_n t}N(t),$$

where we normalize the detrended $N(t)$ to 1.

For sector-level variables, we normalize indices by the measure of variety, $\tilde{i} = i/N(t)$, and define detrended functions as

$$v(\tilde{i}) = e^{\left(\gamma+\theta\frac{1-\beta}{\beta}\right)g_n t}V_i(t), \quad z(\tilde{i}) = z_i(t), \quad e(\tilde{i}) = e(i,t), \quad f(\tilde{i}) = F(i,t),$$

$$p(\tilde{i}) = e^{\gamma g_n t}P(i,t), \quad b(\tilde{i}) = e^{-\frac{1-\beta}{\beta}g_n t}B(i,t), \quad v^f(\tilde{i}) = e^{\gamma g_n t}V^F(i,t).$$

After this normalization, the system becomes stationary, which provides the foundation for characterizing the equilibrium. In what follows, we derive explicit expressions for the seven sector-level variables defined above.

Incumbent's decision $z(\tilde{i})$ and $v(\tilde{i})$. It follows from equation (B.5) that

$$z(\tilde{i}) = \psi\left[\frac{w}{(\lambda^{\theta\frac{1-\beta}{\beta}}-1)\,v(\tilde{i})\,\psi\alpha}\right]^{\frac{\alpha}{\alpha-1}}. \tag{B.34}$$

Since $\tilde{i} = i/N(t)$ and $\dot{N}/N = g_n$, we have the relation $\frac{d\tilde{i}}{dt} = -\tilde{i}\,g_n$. Using the HJB equation (B.10), we can derive

$$\begin{aligned} & [\rho + ((\sigma-1)(1-\gamma)+1)g_n + \delta]\,v(\tilde{i}) \\ = \quad & \beta(1-\beta)^{\frac{1-\beta}{\beta}} \cdot p(\tilde{i})^{\frac{1}{\beta}} \cdot w^{-\frac{1-\beta}{\beta}} \end{aligned}$$

$$-w\left(\frac{z(\tilde{i})}{\psi}\right)^{\frac{1}{\alpha}} + z(\tilde{i})\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right)\cdot v(\tilde{i}) - \tilde{i}g_n v'(\tilde{i}), \tag{B.35}$$

where we have used the Euler equation $r = \rho + \sigma(1-\gamma)g_n$ along the BGP.

Variety of intermediate input $e(\tilde{i})$ and $f(\tilde{i})$. We next characterize the entry dynamics of intermediate firms. For a new entrant, it follows from equation (10) that the expected firm value is given by

$$\begin{aligned}
\mathbb{E}_t[A^{\theta\frac{1-\beta}{\beta}}V_i(t)] &= \left[\frac{\int_0^{F(i,t)} \tilde{A}_i(j,t)^{\theta\frac{1-\beta}{\beta}}dj}{F(i,t)}\right]V_i(t) \\
&= \left[\frac{B(i,t)}{\bar{A}(t)^{(1-\theta)\frac{1-\beta}{\beta}}F(i,t)}\right]V_i(t).
\end{aligned}$$

Define the detrended R&D input for entry as $L_i^E(j,t) = l^E(\tilde{i})\cdot e^{-g_n t}$, then the innovation production function can be written as

$$e(\tilde{i}) = \psi_e l^E(\tilde{i}),$$

and the entrant's decision problem becomes

$$\max_{l^E(\tilde{i})} \quad \psi_e l^E(\tilde{i})\cdot\left(\frac{b(\tilde{i})}{f(\tilde{i})}\right)v(\tilde{i}) - w\cdot l^E(\tilde{i}).$$

The corresponding first-order condition with respect to $l^E(\tilde{i})$ is

$$w = v(\tilde{i})\cdot\psi_e\cdot\left(\frac{b(\tilde{i})}{f(\tilde{i})}\right). \tag{B.36}$$

Finally, the law of motion for the measure of firms $F(i,t)$ is

$$\begin{aligned}
\frac{\partial F(i,t)}{\partial t} &= e(i,t) - \delta F(i,t), \\
\Rightarrow -\tilde{i}g_n\frac{df(\tilde{i})}{d\tilde{i}} &= e(\tilde{i}) - \delta f(\tilde{i}).
\end{aligned} \tag{B.37}$$

Household's decision $p(\tilde{i})$. As we have shown in Lemma B.8, for $\tilde{i}\geq\eta$, $p(\tilde{i}) = y\mathcal{L}'(\tilde{i})$. The equilibrium pricing condition holds for all $\tilde{i}\in(0,1]$:

$$\begin{aligned}
Y(i,t) &= (1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{W(t)}{P(i,t)}\right]^{-\frac{1-\beta}{\beta}}B(i,t) \\
\Rightarrow 1-\mathcal{G}(\mathcal{L}(\tilde{i})) &= (1-\beta)^{\frac{1-2\beta}{\beta}}\left[\frac{w}{p(\tilde{i})}\right]^{-\frac{1-\beta}{\beta}}b(\tilde{i}).
\end{aligned} \tag{B.38}$$

Since $\mathcal{G}(\underline{l}) = 0$ and $\mathcal{L}(\tilde{i}) = \underline{l}$ when $0 < \tilde{i} < \eta$, it follows that

$$1 \quad = \quad (1-\beta)^{\frac{1-2\beta}{\beta}} \left[\frac{w}{p(\tilde{i})}\right]^{-\frac{1-\beta}{\beta}} b(\tilde{i}) \qquad \text{for} \quad \tilde{i} \in (0, \eta). \tag{B.39}$$

Productivity dynamics $b(\tilde{i})$.

We use equation (19) to derive $b(\tilde{i})$. As shown in Lemma B.10, $B(i,t)$ admits the representation $B(i,t) = e^{\frac{1-\beta}{\beta} g_n t} b(\tilde{i})$. Fix any i, since $N(t)$ grows at rate g_n, we have $\tilde{i} = i/N(t) \to 0$ as $t \to \infty$. Denote $b_0 = \lim_{\tilde{i}\to 0} b(\tilde{i})$, then

$$B(i,t) \sim e^{\frac{1-\beta}{\beta} g_n t} b_0 \quad \text{as} \quad t \to \infty.$$

Therefore, the asymptotic growth rate of $B(i,t)$ is $\lim_{t\to\infty} \frac{\dot{B}(i,t)}{B(i,t)} = \frac{1-\beta}{\beta} g_n$. Moreover, the firm-number dynamics in equation (B.37) imply that $\lim_{t\to\infty} e(i,t) = \delta F(i,t)$. Hence, taking limits as $t \to \infty$ in equation (19) gives

$$g_n = \lim_{t\to\infty} \frac{1}{\theta} \frac{\beta}{1-\beta} z_i(t) \left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right) = \frac{1}{\theta} \frac{\beta}{1-\beta} z_0 \left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right), \tag{B.40}$$

where $z_0 \equiv \lim_{\tilde{i}\to 0} z(\tilde{i})$. Then using equation (19) in detrended terms, the law of motion for $b(\tilde{i})$ becomes

$$-\tilde{i} g_n \frac{db(\tilde{i})}{d\tilde{i}} = \left\{ \left[z(\tilde{i}) - z_0\right] \left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right) + \frac{e(\tilde{i})}{f(\tilde{i})} - \delta \right\} b(\tilde{i}). \tag{B.41}$$

Final goods firm $v^f(\tilde{i})$. The aggregate profit of the intermediate producers is:

$$\int_0^{F(i,t)} \Pi_i(j,t) dj = \beta(1-\beta)^{\frac{1-\beta}{\beta}} \cdot P(i,t)^{\frac{1}{\beta}} \cdot W(t)^{-\frac{1-\beta}{\beta}} \cdot B(i,t).$$

The profit flow of the final goods producer is $\frac{1}{1-\beta}$ times the above profit:

$$\Pi(i,t) = \beta(1-\beta)^{\frac{1-2\beta}{\beta}} \cdot P(i,t)^{\frac{1}{\beta}} \cdot W(t)^{-\frac{1-\beta}{\beta}} \cdot B(i,t).$$

So the value of a final-goods firm can be represented by:

$$\begin{aligned} V^F(i,t) &= \int_t^\infty e^{-r(\tau-t)} \beta(1-\beta)^{\frac{1-2\beta}{\beta}} P(i,\tau)^{\frac{1}{\beta}} W(\tau)^{-\frac{1-\beta}{\beta}} B(i,\tau)\, d\tau \\ &= e^{-\gamma g_n t} \int_t^\infty e^{-(r+\gamma g_n)(\tau-t)} \beta(1-\beta)^{\frac{1-2\beta}{\beta}} p(i,\tau)^{\frac{1}{\beta}} w^{-\frac{1-\beta}{\beta}} b(i,\tau)\, d\tau. \end{aligned}$$

Let $V^F(i,t) = e^{-\gamma g_n t} v^f(\tilde{i})$ and $x = i/N(\tau)$. Since $x = \tilde{i}\, e^{-g_n(\tau-t)}$, we have $\tau = \frac{\ln\tilde{i} - \ln x}{g_n} + t$, and

$$v^f(\tilde{i}) \quad = \quad \int_{\tilde{i}}^0 e^{-(r+\gamma g_n)\frac{\ln\tilde{i}-\ln x}{g_n}} \beta(1-\beta)^{\frac{1-2\beta}{\beta}} p(x)^{\frac{1}{\beta}} w^{-\frac{1-\beta}{\beta}} b(x)\, d\left(\frac{\ln\tilde{i} - \ln x}{g_n} + t\right)$$

$$
\begin{aligned}
&= \int_0^{\tilde{i}} \frac{e^{-(\frac{r}{g_n}+\gamma)(\ln \tilde{i}-\ln x)}}{x g_n}\, \beta(1-\beta)^{\frac{1-2\beta}{\beta}}\, p(x)^{\frac{1}{\beta}}\, w^{-\frac{1-\beta}{\beta}}\, b(x)\, dx \\
&= \int_0^{\tilde{i}} \frac{x^{(\frac{r}{g_n}+\gamma-1)}}{\tilde{i}^{(\frac{r}{g_n}+\gamma)} g_n}\, \beta(1-\beta)^{\frac{1-2\beta}{\beta}}\, p(x)^{\frac{1}{\beta}}\, w^{-\frac{1-\beta}{\beta}}\, b(x)\, dx \\
&= \int_0^{\tilde{i}} \frac{x^{\left[\frac{\rho}{g_n}+\sigma(1-\gamma)+\gamma-1\right]}}{\tilde{i}^{\left[\frac{\rho}{g_n}+\sigma(1-\gamma)+\gamma\right]} g_n}\, \beta(1-\beta)^{\frac{1-2\beta}{\beta}}\, p(x)^{\frac{1}{\beta}}\, w^{-\frac{1-\beta}{\beta}}\, b(x)\, dx. \qquad \text{(B.42)}
\end{aligned}
$$

B.4 Proofs for Proposition 1

Proof of Proposition 1:

To prove Proposition 1, we show that a BGP can be characterized by a pair of functions $\{v(\tilde{i}), \mathcal{L}(\tilde{i})\}_{\tilde{i}\in(0,1]}$ and a set of parameters $\{w, \eta, y, g_n\}$. The argument proceeds by verifying that these objects jointly satisfy the following system of conditions:

(i) The ordinary differential equation for $v(\tilde{i})$ together with its boundary condition;

(ii) The ordinary differential equation for $\mathcal{L}(\tilde{i})$ together with its boundary condition;

(iii) The labor market clearing condition;

(iv) The aggregate resource constraint;

(v) The entry condition of final producers;

(vi) The price normalization.

Establishing that $\{v(\tilde{i}), \mathcal{L}(\tilde{i}), w, \eta, y, g_n\}$ satisfy all six conditions completes the proof of the proposition.

Step 1. Derivation of the ODE for $v(\tilde{i})$ and its boundary condition.

We begin by deriving the differential equation satisfied by $v(\tilde{i})$. From the detrended entry condition in equation (B.36),

$$w \;=\; v(\tilde{i}) \cdot \psi_e \cdot \frac{b(\tilde{i})}{f(\tilde{i})},$$

taking logs and differentiating with respect to $\tilde{i}$ yields

$$\frac{v'(\tilde{i})}{v(\tilde{i})} = \frac{f'(\tilde{i})}{f(\tilde{i})} - \frac{b'(\tilde{i})}{b(\tilde{i})}.$$

Substituting $\frac{f'(\tilde{i})}{f(\tilde{i})}$ from equation (B.37) and $\frac{b'(\tilde{i})}{b(\tilde{i})}$ from equation (B.41) gives

$$v'(\tilde{i}) = \left[z(\tilde{i}) - z_0\right] \cdot \left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right) \frac{v(\tilde{i})}{\tilde{i} g_n}. \qquad \text{(B.43)}$$

Using equation (B.40), we can rewrite the above equation as

$$v'(\tilde{i}) = \theta \frac{1-\beta}{\beta} \cdot \left[\tfrac{z(\tilde{i})}{z_0} - 1\right] \cdot \frac{v(\tilde{i})}{\tilde{i}}. \tag{B.44}$$

Since $z(\tilde{i})$ satisfies equation (B.34), by denoting $v_0 = \lim_{\tilde{i} \to 0} v(\tilde{i})$, we can derive $\frac{z(\tilde{i})}{z_0} = \left(\frac{v(\tilde{i})}{v_0}\right)^{\frac{\alpha}{1-\alpha}}$. Substituting this equation into equation (B.44), we arrive at

$$v'(\tilde{i}) = \theta \frac{1-\beta}{\beta} \cdot \left[\left(\tfrac{v(\tilde{i})}{v_0}\right)^{\frac{\alpha}{1-\alpha}} - 1\right] \frac{v(\tilde{i})}{\tilde{i}}. \tag{B.45}$$

Next, equations (B.40) and equation (B.34) allow us to eliminate z_0 and express v_0 as a function of g_n:

$$v_0 = h(g_n) = \frac{w\left(\theta \frac{1-\beta}{\beta}\right)^{\frac{1-\alpha}{\alpha}}}{\left[\left(\lambda^{\theta \frac{1-\beta}{\beta}} - 1\right)\psi\right]^{1/\alpha} \alpha} \, {g_n}^{\frac{1-\alpha}{\alpha}}.$$

Substituting this expression into equation (B.45) yields

$$v'(\tilde{i}) = \theta \frac{1-\beta}{\beta} \cdot \left[\left(\tfrac{v(\tilde{i})}{h(g_n)}\right)^{\frac{\alpha}{1-\alpha}} - 1\right] \frac{v(\tilde{i})}{\tilde{i}},$$

which corresponds exactly to equation (22) in the proposition.

Finally, we derive the boundary condition. We define $A(i,t)$ as the average firm productivity in sector i at time t:

$$A(i,t) = \left[\frac{\int_0^{F(i,t)} \tilde{A}_i(j,t)^{\theta \frac{1-\beta}{\beta}} dj}{F(i,t)}\right]^{\frac{1}{\theta} \frac{\beta}{1-\beta}}. \tag{B.46}$$

By assumption, every intermediate-good producers in the newest variety $N(t)$ has productivity $\tilde{A}_{N(t)}(j,t) = \zeta \bar{A}(t)$. Substituting this equation into equation (B.46) yields $A(N(t),t) = \zeta \bar{A}(t)$. Defining the detrended average firm productivity in a given sector as $a(\tilde{i}) = e^{-g_n t} A(i,t)$, we obtain $a(1) = \zeta$. It follows from equation (10) that

$$A(i,t)^{\theta \frac{1-\beta}{\beta}} = \frac{B(i,t)}{\bar{A}(t)^{(1-\theta)\frac{1-\beta}{\beta}} F(i,t)},$$

which implies $a(\tilde{i})^{\theta \frac{1-\beta}{\beta}} = \frac{b(\tilde{i})}{f(\tilde{i})}$. Substituting this into equation (17), we obtain

$$w = v(\tilde{i}) \cdot \psi_e \cdot a(\tilde{i})^{\theta \frac{1-\beta}{\beta}}, \tag{B.47}$$

and using $a(1) = \zeta$, it follows that

$$v(1) = \frac{w}{\psi_e \zeta^{\theta \frac{1-\beta}{\beta}}}.$$

This completes the derivation of equation (22) and its boundary condition.

Step 2. Derivation of the ODE for $\mathcal{L}(\tilde{i})$ and its boundary condition.
The detrended HJB equation (B.35) for an incumbent is

$$\begin{aligned}&\big[\rho+(\sigma-1)(1-\gamma)g_n+g_n+\delta\big]v(\tilde{i})\\ =\ &\beta(1-\beta)^{\frac{1-\beta}{\beta}}\cdot p(\tilde{i})^{\frac{1}{\beta}}\cdot w^{-\frac{1-\beta}{\beta}}-w\left[\frac{w}{\big(\lambda^{\theta\frac{1-\beta}{\beta}}-1\big)\,v(\tilde{i})\,\psi\alpha}\right]^{\frac{1}{\alpha-1}}\\ &+z(\tilde{i})\big(\lambda^{\theta\frac{1-\beta}{\beta}}-1\big)\,v(\tilde{i})-\tilde{i}g_nv'(\tilde{i}).\end{aligned}$$

Using equation (B.43) from **Step 1** to substitute $v'(\tilde{i})$ and using equation (B.40) to eliminate z_0, the above HJB equation is simplified to

$$\begin{aligned}\big[\rho+\delta+\big((\sigma-1)(1-\gamma)+1-\theta\tfrac{1-\beta}{\beta}\big)g_n\big]v(\tilde{i})&=\beta(1-\beta)^{\frac{1-\beta}{\beta}}\cdot p(\tilde{i})^{\frac{1}{\beta}}\cdot w^{-\frac{1-\beta}{\beta}}\\ &\quad-w\left[\frac{w}{\big(\lambda^{\theta\frac{1-\beta}{\beta}}-1\big)\,v(\tilde{i})\,\psi\alpha}\right]^{\frac{1}{\alpha-1}}.\end{aligned}$$

Rearranging gives the closed-form pricing function

$$\begin{aligned}p(\tilde{i})\ &=\ \frac{w^{1-\beta}}{\beta^{\beta}(1-\beta)^{(1-\beta)}}\cdot\Bigg\{\Big[\rho+\delta+\Big((\sigma-1)(1-\gamma)+1-\theta\frac{1-\beta}{\beta}\Big)g_n\Big]v(\tilde{i})\\ &\qquad+w\cdot\left[\frac{w}{\big(\lambda^{\theta\frac{1-\beta}{\beta}}-1\big)v(\tilde{i})\cdot\psi\alpha}\right]^{\frac{1}{\alpha-1}}\Bigg\}^{\beta}.\end{aligned}\tag{B.48}$$

From Lemma B.5, the demand system implies $p(\tilde{i})=y\mathcal{L}'(\tilde{i})$ for $\tilde{i}\in[\eta,1]$, so that the expression above directly characterizes the ODE for $\mathcal{L}(\tilde{i})$ in terms of $v(\tilde{i})$ and the model primitives:

$$\begin{aligned}\mathcal{L}'(\tilde{i})\ &=\ \frac{w^{1-\beta}}{y\beta^{\beta}(1-\beta)^{(1-\beta)}}\cdot\Bigg\{\Big[\rho+\delta+\Big((\sigma-1)(1-\gamma)+1-\theta\frac{1-\beta}{\beta}\Big)g_n\Big]v(\tilde{i})\\ &\qquad+w\cdot\left[\frac{w}{\big(\lambda^{\theta\frac{1-\beta}{\beta}}-1\big)v(\tilde{i})\cdot\psi\alpha}\right]^{\frac{1}{\alpha-1}}\Bigg\}^{\beta}.\end{aligned}\tag{B.49}$$

At last, from Lemma B.8, the boundary condition is $\mathcal{L}(\eta)=\underline{l}$. And if $\tilde{i}<\eta$, $\mathcal{L}(\tilde{i})=\underline{l}$, which pines down $\mathcal{L}(\tilde{i})$.

Having solved for $v(\tilde{i})$ and $\mathcal{L}(\tilde{i})$ in **Step 1** and **Step 2**, we can then recover $p(\tilde{i})$ from equation (B.48), $b(\tilde{i})$ from equation (B.38), $v^f(\tilde{i})$ from equation (B.42), $f(\tilde{i})$ from equation (B.36), and

$e(\tilde{i})$ from equation (B.37), taking $\{w, \eta, y, g_n\}$ as given. These detrended objects are then directly used in the subsequent steps.

Step 3. Labor market clearing condition.

Labor is demanded by production ($\int_0^{N(t)} \int_0^{F(i,t)} L_i(j,t)djdi$), by incumbent R&D ($\int_0^{N(t)} \int_0^{F(i,t)} L_i^I djdi$), by entry R&D ($L^E(t)$), by luxury service ($L^X(t)$), and by idea production ($L^X(t)$). So the labor market clearing condition is

$$1 = \int_0^{N(t)} \left[\int_0^{F(i,t)} L_i(j,t) + L_i^I(j,t)dj \right] di + L^E(t) + L^X(t) + L^N(t).$$

Specifically, along the BGP, this condition can be re-expressed as

$$1 \ = \ \int_0^1 \left[l(\tilde{i}) + l^I(\tilde{i}) \right] d\tilde{i} \ + \ L^E \ + \ L^X \ + \ L^N, \tag{B.50}$$

where:

(i) $l(\tilde{i}) \equiv e^{g_n t} L(i,t)$ is defined as the detrended labor demand in final-good sector $\tilde{i}$. The expression for $L(i,t) = \int_0^{F(i,t)} L_i(j,t)dj$ is given in equation (12), with a detrended version

$$l(\tilde{i}) = \left[\frac{w}{(1-\beta)p(\tilde{i})} \right]^{-\frac{1}{\beta}} b(\tilde{i}). \tag{B.51}$$

(ii) $l^I(\tilde{i}) \equiv e^{g_n t} \int_0^{F(i,t)} L_i^I(j,t)dj$ is defined as the incumbents' R&D demand in final-good sector $\tilde{i}$. The expression for $L_i^I(j,t)$ is given in equation (B.9), which leads to

$$l^I(\tilde{i}) = \left[\frac{w}{(\lambda^{\theta \frac{1-\beta}{\beta}} - 1) v(\tilde{i}) \psi \alpha} \right]^{\frac{1}{\alpha - 1}} \cdot b(\tilde{i}). \tag{B.52}$$

(iii) L^E is the entrants' R&D labor demand defined in equation (B.15). Expressing it in detrended terms yields

$$L^E = \int_0^1 \frac{e(\tilde{i})}{\psi_e} d\tilde{i} + \frac{f(1)g_n}{\psi_e}. \tag{B.53}$$

(iv) L^X denotes labor demand in the luxury service sector, which is pinned down by the luxury service market clearing condition. As we have shown in Lemma B.7, the aggregate demand for luxury service is

$$\frac{\int_{\mathcal{L}(1)}^{\infty} \big(l - \mathcal{L}(1)\big) Y(t) d\mathcal{G}(l)}{P^X(t)},$$

so we have

$$
\begin{aligned}
L^X &= \frac{\int_{\mathcal{L}(1)}^{\infty} \big(l - \mathcal{L}(1)\big) Y(t) d\mathcal{G}(l)}{\bar{A}(t)^{1-\gamma} P^X(t)} \\
&= \frac{\int_{\mathcal{L}(1)}^{\infty} \big(l - \mathcal{L}(1)\big) Y(t) d\mathcal{G}(l)}{W(t)} \\
&= \frac{y \int_{\mathcal{L}(1)}^{\infty} \big(l - \mathcal{L}(1)\big) d\mathcal{G}(l)}{w},
\end{aligned}
\tag{B.54}
$$

where the second equality follows from the expression for $P^X(t)$ in equation (B.29).

(v) L^N is the labor demand in the idea production sector. From equation (20), we have

$$
L^N = \frac{g_n}{\phi}. \tag{B.55}
$$

In equilibrium, equation (B.50) pins down the wage w: the left–hand side is total labor supply (normalized to one), while the right–hand side is total labor demand expressed in detrended variables.

Step 4. Aggregate resource constraint.

The aggregate resource constraint is

$$
\underbrace{Y(t)}_{\text{aggregate output}} = W(t) \underbrace{\left[\int_0^{N(t)} L(i,t) di + L^X(t)\right]}_{\text{total production labor}} + \underbrace{\int_0^{N(t)} \left[\Pi(i,t) + \int_0^{F(i,t)} \Pi_i(j,t) dj\right] di}_{\text{total production profit}}, \tag{B.56}
$$

which states that aggregate output must equal the sum of labor income from production and profits. From the equilibrium condition of the luxury service producer, we have

$$
X(t) P^X(t) = W(t) L^X(t).
$$

Then equation (B.56) becomes

$$
\int_0^{N(t)} P(i,t) Y(i,t) di = W(t) \int_0^{N(t)} L(i,t) di + \int_0^{N(t)} \left[\Pi(i,t) + \int_0^{F(i,t)} \Pi_i(j,t) dj\right] di.
$$

In Appendix B.2.1, we have shown that

$$
\frac{\Pi(i,t)}{P(i,t)Y(i,t)} = \beta \quad \text{and} \quad \frac{\int_0^{F(i,t)} \Pi_i(j,t) dj}{P(i,t)Y(i,t)} = \beta(1-\beta),
$$

which implies

$$
(1-\beta)^2 \int_0^{N(t)} P(i,t) Y(i,t) di = W(t) \int_0^{N(t)} L(i,t) di. \tag{B.57}
$$

Using the market-clearing condition of the final-goods producer, we derive

$$\begin{aligned}
\int_0^{N(t)} P(i,t)Y(i,t)di &= \int_{\underline{l}}^{\infty}\left[\int_0^{N(t)} P(i,t)C_l(i,t)\,di\right]d\mathcal{G}(l). \\
&= \int_{\underline{l}}^{\bar{l}}\int_0^{N(t)\mathcal{L}^{-1}(l)} P(i,t)did\mathcal{G}(l) + \int_{\bar{l}}^{\infty}\int_0^{N(t)} P(i,t)did\mathcal{G}(l) \\
&= \int_{\underline{l}}^{\mathcal{L}(1)} Y(t)ld\mathcal{G}(l) + \int_{\mathcal{L}(1)}^{\infty} Y(t)\mathcal{L}(1)d\mathcal{G}(l) \\
&= Y(t)\left[1-\int_{\mathcal{L}(1)}^{\infty}\big(l-\mathcal{L}(1)\big)\,d\mathcal{G}(l)\right],
\end{aligned}$$

where we have used Lemma B.7 and equation (B.28). Substituting the last expression into equation (B.57), we have the detrended version of the resource constraint:

$$\begin{aligned}
(1-\beta)^2 Y(t)\left[1-\int_{\mathcal{L}(1)}^{\infty}\big(l-\mathcal{L}(1)\big)\,d\mathcal{G}(l)\right] &= W(t)\int_0^{N(t)} L(i,t)di \\
\Rightarrow y(1-\beta)^2\left[1-\int_{\mathcal{L}(1)}^{\infty}\big(l-\mathcal{L}(1)\big)\,d\mathcal{G}(l)\right] &= w\int_0^1 l(\tilde{i})\,d\tilde{i}. \qquad \text{(B.58)}
\end{aligned}$$

Step 5. Entry condition of intermediate goods producers.

The inventor's optimization problem in equation (21) implies

$$w = v^f(1)\phi, \qquad \text{(B.59)}$$

where $v^f(\tilde{i})$ denotes the detrended value of the newest final–good firm, defined in equation (B.42).

Step 6. Price normalization.

First, we show that the price normalization implies $C(t) = Y(t)$. We define the price of final consumption bundle as $P^C(t)$, which satisfies

$$P^C(t)C(t) = \int_{\underline{l}}^{\infty}\left[\int_0^{N(t)} P(i,t)C_l(i,t)\,di + P^X(t)X_l(t)\right]d\mathcal{G}(l). \qquad \text{(B.60)}$$

Since this price is normalized to one, i.e. $P^C(t) = 1$, we have

$$\begin{aligned}
C(t) &= \int_{\underline{l}}^{\infty}\left[\int_0^{N(t)} P(i,t)C_l(i,t)\,di + P^X(t)X_l(t)\right]d\mathcal{G}(l) \\
&= \int_{\underline{l}}^{\infty}\left[\int_0^{N(t)} P(i,t)C_l(i,t)\,di\right]d\mathcal{G}(l) + \int_{\bar{l}}^{\infty} P^X(t)X_l(t)\,d\mathcal{G}(l). \qquad \text{(B.61)}
\end{aligned}$$

The market-clearing condition for the luxury services implies

$$\int_{\bar{l}}^{\infty} P^X(t)X_l(t)\,d\mathcal{G}(l) = P^X(t)X(t),$$

and the market-clearing condition for the final goods implies

$$\int_0^{N(t)} P(i,t)Y(i,t)di \quad = \quad \int_{\underline{l}}^{\infty} \left[\int_0^{N(t)} P(i,t)C_l(i,t)\, di\right] d\mathcal{G}(l). \tag{B.62}$$

So equation (B.61) simplifies to $C(t) = \int_0^{N(t)} P(i,t)Y(i,t)di + P^X(t)X(t) = Y(t)$.

Next, we use $C(t) = Y(t)$ to rewrite $Y(t)$ as

$$\begin{aligned} Y(t) \quad &= \quad C(t) = \int_{\underline{l}}^{\infty} C_l(t) d\mathcal{G}(l) \\ &= \quad \int_{\underline{l}}^{\infty} \left[\int_0^{N(t)} i^{-\gamma} C_l(i,t)di + \min_{\{i\in(0,N_t]\}} \{(C_l(i,t))\} \cdot X_l(t)\right] d\mathcal{G}(l). \end{aligned} \tag{B.63}$$

The market-clearing condition for the luxury service implies that

$$\int_{\underline{l}}^{\infty} \min_{\{i\in(0,N_t]\}} \{(C_l(i,t))\} \cdot X_l(t) d\mathcal{G}(l) = X(t) = \bar{A}(t)^{1-\gamma} L^X(t),$$

and it follows from Lemma B.7 that

$$\begin{aligned} \int_{\underline{l}}^{\infty} \left[\int_0^{N(t)} i^{-\gamma} C_l(i,t)di\right] d\mathcal{G}(l) \quad &= \quad \int_{\underline{l}}^{\mathcal{L}(1)} \left[\int_0^{\mathcal{L}^{-1}(l)N(t)} i^{-\gamma} di\right] d\mathcal{G}(l) + \int_{\mathcal{L}(1)}^{\infty} \left[\int_0^{N(t)} i^{-\gamma} di\right] d\mathcal{G}(l) \\ &= \quad \int_{\underline{l}}^{\mathcal{L}(1)} \frac{[\mathcal{L}^{-1}(l)N(t)]^{1-\gamma}}{1-\gamma} d\mathcal{G}(l) + \int_{\mathcal{L}(1)}^{\infty} \frac{N(t)^{1-\gamma}}{1-\gamma} d\mathcal{G}(l). \end{aligned}$$

Substituting these two relations into equation (B.63) yields

$$Y(t) \quad = \quad \int_{\underline{l}}^{\mathcal{L}(1)} \frac{[\mathcal{L}^{-1}(l)N(t)]^{1-\gamma}}{1-\gamma} d\mathcal{G}(l) + \int_{\mathcal{L}(1)}^{\infty} \frac{N(t)^{1-\gamma}}{1-\gamma} d\mathcal{G}(l) + \bar{A}(t)^{1-\gamma} L^X(t). \tag{B.64}$$

The detrended version of equation (B.64) is

$$y \quad = \quad \int_{\underline{l}}^{\mathcal{L}(1)} \frac{[\mathcal{L}^{-1}(l)]^{1-\gamma}}{1-\gamma} d\mathcal{G}(l) + \int_{\mathcal{L}(1)}^{\infty} \frac{1}{1-\gamma} d\mathcal{G}(l) + \frac{g_n}{\phi}, \tag{B.65}$$

where we have substituted the expression of L^X in equation (B.55).

Finally, we verify Conjecture 1 proposed in Appendix B.3.1, which states that $P_l(t)$ is constant for each time t and for all households. For households with $l < \mathcal{L}(1)$, we have

$$P_l(t) \quad = \quad \frac{\int_0^{N(t)} P(i,t)C_l(i,t)\, di}{\int_0^{N(t)} i^{-\gamma} C_l(i,t)\, di} = \frac{\int_0^{\mathcal{L}^{-1}(l)} p(\tilde{i})\, d\tilde{i}}{\int_0^{\mathcal{L}^{-1}(l)} \tilde{i}^{-\gamma}\, d\tilde{i}}. \tag{B.66}$$

For households with $l > \mathcal{L}(1)$, we have

$$P_l(t) \quad = \quad \frac{\int_0^{N(t)} P(i,t) C_l(i,t)\, di + P^X(t) X_l(t)}{\int_0^{N(t)} i^{-\gamma} C_l(i,t)\, di + X_l(t)} = \frac{\int_0^1 p(\tilde{i})\, d\tilde{i} + y(l - \bar{l})}{\int_0^1 \tilde{i}^{-\gamma}\, d\tilde{i} + \frac{y(l-\bar{l})}{w}}, \tag{B.67}$$

where the second equality follows from the expression for $X_l(t)$ in equation (B.22) and the expression for $P^X(t)$ in equation (B.29).

Combining Steps 1 through 6, we establish that the BGP can be represented by a pair of functions $\{v(\tilde{i}), \mathcal{L}(\tilde{i})\}_{\tilde{i} \in (0,1]}$ and a set of parameters $\{w, \eta, y, g_n\}$. Together, these conditions characterize the BGP and hence the proof of Proposition 1 is completed. ■

B.5 Proofs for Proposition 2

Before formally proving the proposition, we first establish the condition of ζ, which is summarized in the following assumption and lemma.

Assumption 2 *We assume ζ is relatively small, such as*

$$\zeta < \left\{ \frac{\left[\left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1 \right) \psi \right]^{1/\alpha} \alpha}{\psi_e \left(\theta^{\frac{1-\beta}{\beta}} \right)^{\frac{1-\alpha}{\alpha}} g_n^{\frac{1-\alpha}{\alpha}}} \right\}^{\frac{1}{\theta} \frac{\beta}{1-\beta}}. \tag{B.68}$$

Lemma B.11 *Under the Assumption 2, the average firm productivity in a given sector $a(\tilde{i})$ satisfies*

$$a(1) < a(0), \;\; \text{and } v(1) > v_0 \equiv \lim_{\tilde{i} \to 0} v(\tilde{i}),$$

where $a(\tilde{i})$ is defined as the detrended version of equation (B.46), while $a(0) \equiv \lim_{\tilde{i} \to 0} a(\tilde{i})$.

Proof of Lemma B.11:

Using the expressions for $v(1)$ and v_0 provided in Proposition 1, we obtain

$$\frac{v(1)}{v_0} = \frac{\left[\left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1 \right) \psi \right]^{1/\alpha} \alpha}{\psi_e \left(\theta^{\frac{1-\beta}{\beta}} \right)^{\frac{1-\alpha}{\alpha}} g_n^{\frac{1-\alpha}{\alpha}} \zeta^{\theta^{\frac{1-\beta}{\beta}}}} > 1.$$

Then, from equation (B.47),

$$\frac{a(1)}{a(0)} = \left[\frac{v_0}{v(1)} \right]^{\frac{1}{\theta} \frac{\beta}{1-\beta}} < 1.$$

■

Proof of Proposition 2:

We begin by establishing the monotonicity of $v(\tilde{i})$, then we show the monotonicity of $z(\tilde{i})$, $p(\tilde{i})$, and $b(\tilde{i})$, respectively.

Step 1. Monotonicity of $v(\tilde{i})$.

We first show that $\lim_{\tilde{i}\to 0} v'(\tilde{i}) = 0$, and then establish that $v'(\tilde{i}) > 0$ for all $\tilde{i} \in (0,1]$. From Proposition 1, $v(\tilde{i})$ satisfies the ODE

$$v'(\tilde{i}) = \theta\frac{1-\beta}{\beta}\left[\left(\frac{v(\tilde{i})}{v_0}\right)^{\frac{\alpha}{1-\alpha}} - 1\right]\frac{v(\tilde{i})}{\tilde{i}}, \qquad \tilde{i} \in (0,1]. \tag{B.69}$$

To compute $\lim_{\tilde{i}\to 0} v'(\tilde{i})$, we apply L'Hospital's rule:

$$\begin{aligned} \lim_{\tilde{i}\to 0} v'(\tilde{i}) &= \theta\frac{1-\beta}{\beta}\cdot\lim_{\tilde{i}\to 0}\frac{\alpha}{1-\alpha}\left(\frac{v(\tilde{i})}{v_0}\right)^{\frac{\alpha}{1-\alpha}} v'(\tilde{i}) \\ &= \theta\frac{1-\beta}{\beta}\cdot\frac{\alpha}{1-\alpha}\lim_{\tilde{i}\to 0} v'(\tilde{i}). \end{aligned}$$

Since $\theta\frac{1-\beta}{\beta}\frac{\alpha}{1-\alpha} \neq 1$ in general, it must be that

$$\lim_{\tilde{i}\to 0} v'(\tilde{i}) = 0.$$

Given this condition, the right-hand side of equation (B.69) is bounded and satisfies Lipschitz condition; therefore, the solution to the ODE is unique. Now we prove v is monotonically increasing by contradiction. Suppose there exists $k \in (0,1]$ such that $v'(k) \leq 0$. Two cases arise:

- If $v'(k) = 0$, then $v(\tilde{i}) = v_0$ solves equation (B.69). By uniqueness of solutions, this would imply $v(1) = v_0$, which is contradicted with Lemma B.11.
- If $v'(k) < 0$, note that $v'(\tilde{i}) > 0$ if and only if $v(\tilde{i}) > v_0$. Thus $v(k) < v_0$. Since $v(1) > v_0$ and $v(\tilde{i})$ is continuous, there must exist some $k' \in (k,1)$ with $v(k') = v_0$, which implies $v'(k') = 0$, still contradicted.

Therefore $v'(\tilde{i}) > 0$ for all $\tilde{i} \in (0,1]$, i.e. $v(\tilde{i})$ is strictly increasing.

Step 2. Monotonicity of other variables.

- *Innovation intensity $z(\tilde{i})$ and price $p(\tilde{i})$:* Equation (B.34) and equation (B.48)

$$\begin{aligned} z(\tilde{i}) &= \psi\left[\frac{w}{\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right)v(\tilde{i})\cdot\psi\alpha}\right]^{\frac{\alpha}{\alpha-1}} \\ p(\tilde{i}) &= \frac{w^{1-\beta}}{\beta^{\beta}(1-\beta)^{(1-\beta)}}\cdot\Bigg\{\left[\rho+\delta+\left((\sigma-1)(1-\gamma)+1-\theta\frac{1-\beta}{\beta}\right)g_n\right]v(\tilde{i}) \end{aligned}$$

$$\left. +w \cdot \left[\frac{w}{\left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1\right) v(\tilde{i}) \cdot \psi\alpha} \right]^{\frac{1}{\alpha-1}} \right\}^{\beta}.$$

implies that $z(\tilde{i})$ and $p(\tilde{i})$ increase with $v(\tilde{i})$. Given $v(\tilde{i})$ is strictly increasing, so are the two functions.

- *Sectoral productivity* $b(\tilde{i})$*:* For $\tilde{i} \in [\eta, 1]$, from equation (B.38),

$$\begin{aligned} b(\tilde{i}) &= \frac{w^{\frac{1-\beta}{\beta}} p(\tilde{i})^{-\frac{1-\beta}{\beta}} (1 - \mathcal{G}(\mathcal{L}(\tilde{i})))}{(1-\beta)^{\frac{1-2\beta}{\beta}}} \\ \Rightarrow b'(\tilde{i}) &= \frac{w^{\frac{1-\beta}{\beta}}}{(1-\beta)^{\frac{1-2\beta}{\beta}}} \cdot \left[-\frac{1-\beta}{\beta} p(\tilde{i})^{-\frac{1}{\beta}} p'(\tilde{i}) (1 - \mathcal{G}(\mathcal{L}(\tilde{i}))) - \mathcal{G}'(\mathcal{L}(\tilde{i})) \mathcal{L}'(\tilde{i}) \right] < 0. \end{aligned}$$

The last inequality comes from the facts that $\mathcal{L}' > 0$. For $\tilde{i} \in (0, \eta)$, from equation (B.39),

$$b(\tilde{i}) = \frac{w^{\frac{1-\beta}{\beta}} p(\tilde{i})^{-\frac{1-\beta}{\beta}}}{(1-\beta)^{\frac{1-2\beta}{\beta}}},$$

which implies that $b(\tilde{i})$ is strictly decreasing give that $p(\tilde{i})$ is strictly increasing.

■

Finally, we verify Conjecture 2 proposed in Appendix B.3.1. In particular, along the BGP, the equilibrium price can be written as

$$P(i,t) = e^{-\gamma g_n t} p(\tilde{i}), \qquad \tilde{i} = \frac{i}{N(t)}.$$

Fix any date t. Since $p(\tilde{i})$ is strictly increasing in $\tilde{i}$, it follows immediately that $P(i,t)$ is strictly increasing in i. Hence, Conjecture 2 holds in equilibrium, thereby completing the proof.

B.6 Proofs for Proposition 3

Proof of Proposition 3:

For notational convenience, we use superscripts 1 and 2 to denote variables corresponding to the equilibrium outcomes under the income distributions $\mathcal{G}^1$ and $\mathcal{G}^2$, respectively. We proceed in two steps: (i) we show that $\frac{z^2(\tilde{i})}{z^1(\tilde{i})}$ is strictly increasing in $\tilde{i}$; (ii) we demonstrate that $\frac{z^2(\tilde{i})}{z^1(\tilde{i})} \leq 1$.

Step 1. $\frac{z^2(\tilde{i})}{z^1(\tilde{i})}$ **is strictly increasing in** $\tilde{i}$**.**

From equation (B.34), we can express

$$\begin{aligned}\frac{z^2(\tilde{i})}{z^1(\tilde{i})} &= \left[\frac{v^2(\tilde{i})}{v^1(\tilde{i})}\right]^{\frac{\alpha}{1-\alpha}} \cdot \left(\frac{w^1}{w^2}\right)^{\frac{\alpha}{1-\alpha}} \\ &= \left[\frac{\hat{v}^2(\tilde{i})}{\hat{v}^1(\tilde{i})}\right]^{\frac{\alpha}{1-\alpha}} \cdot \left(\frac{v_0^2}{v_0^1}\right)^{\frac{\alpha}{1-\alpha}} \cdot \left(\frac{w^1}{w^2}\right)^{\frac{\alpha}{1-\alpha}},\end{aligned}$$

where we use $\hat{v}(\tilde{i}) = v(\tilde{i})/v_0$ to represent the value of $v(\tilde{i})$ relative to v_0. Hence, $\frac{z^2(\tilde{i})}{z^1(\tilde{i})}$ is strictly increasing in $\tilde{i}$ if and only if $\frac{\hat{v}^2(\tilde{i})}{\hat{v}^1(\tilde{i})}$ is strictly increasing in $\tilde{i}$.

According to equation (B.69), $\hat{v}(\tilde{i})$ satisfies the ODE

$$\hat{v}'(\tilde{i}) = \theta\frac{1-\beta}{\beta} \cdot \frac{\left[\hat{v}(\tilde{i})^{\frac{\alpha}{1-\alpha}} - 1\right]\hat{v}(\tilde{i})}{\tilde{i}}, \tag{B.70}$$

with boundary condition

$$\hat{v}(1) = \frac{v(1)}{v_0} = \frac{\left[\left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1\right)\psi\right]^{1/\alpha}\alpha}{\psi_e\left(\theta^{\frac{1-\beta}{\beta}}\right)^{\frac{1-\alpha}{\alpha}} g_n^{\frac{1-\alpha}{\alpha}}\zeta^{\theta^{\frac{1-\beta}{\beta}}}}.$$

Since $\hat{v}(1)$ is decreasing in g_n and $g_n^1 > g_n^2$, it follows that $\hat{v}^2(1) > \hat{v}^1(1)$. Moreover, by uniqueness and continuity of the ODE solution, and $\lim_{\tilde{i}\to 0}\hat{v}^1(\tilde{i}) = \lim_{\tilde{i}\to 0}\hat{v}^2(\tilde{i}) = 1$, we obtain

$$\hat{v}^2(\tilde{i}) > \hat{v}^1(\tilde{i}) > 1 \qquad \text{for all } 0 < \tilde{i} \le 1.$$

Finally, from equation (B.70),

$$\frac{\hat{v}^{2\prime}(\tilde{i})}{\hat{v}^{1\prime}(\tilde{i})} = \frac{\left[\hat{v}^2(\tilde{i})^{\frac{\alpha}{1-\alpha}} - 1\right]\hat{v}^2(\tilde{i})}{\left[\hat{v}^1(\tilde{i})^{\frac{\alpha}{1-\alpha}} - 1\right]\hat{v}^1(\tilde{i})} > \frac{\hat{v}^2(\tilde{i})}{\hat{v}^1(\tilde{i})},$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{i}}\left(\frac{\hat{v}^2(\tilde{i})}{\hat{v}^1(\tilde{i})}\right) = \frac{\hat{v}^{2\prime}(\tilde{i})\hat{v}^1(\tilde{i}) - \hat{v}^{1\prime}(\tilde{i})\hat{v}^2(\tilde{i})}{\left[\hat{v}^1(\tilde{i})\right]^2} > 0.$$

Step 2. $\frac{z^2(\tilde{i})}{z^1(\tilde{i})} \le 1$**.**

Observe that

$$z(1) = \psi\left[\frac{w}{\left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1\right)v(1)\cdot\psi\alpha}\right]^{\frac{\alpha}{\alpha-1}} = \psi\left[\frac{\psi_e\,\zeta^{\theta^{\frac{1-\beta}{\beta}}}}{\left(\lambda^{\theta^{\frac{1-\beta}{\beta}}} - 1\right)\psi\alpha}\right]^{\frac{\alpha}{\alpha-1}},$$

which is independent of equilibrium outcomes. Hence $z^2(1) = z^1(1)$. Since $\frac{z^2(\tilde{i})}{z^1(\tilde{i})}$ is strictly increasing in $\tilde{i}$, it follows that

$$\frac{z^2(\tilde{i})}{z^1(\tilde{i})} \leq 1 \quad \text{for all } \tilde{i}.$$

Combining Steps 1 and 2, we conclude that if $g_n^1 > g_n^2$, then

$$\log(z^1(\tilde{i})) - \log(z^2(\tilde{i})) \geq 0 \quad \text{for all } \tilde{i},$$

and this difference is strictly decreasing in $\tilde{i}$. ■

B.7 Proofs for Proposition 4

Proof of Proposition 4:

The proof of this proposition is already contained in Appendix B.3.3 and in the proof of Lemma B.10. Here we do not repeat the proof. ■

C Appendix: Alternative Model Specification

This section examines the robustness of the main results under different model specifications. While we modify several assumptions of the baseline framework, the core mechanisms of the model remain unchanged. The main qualitative results continue to hold under these alternative settings.

C.1 Extension: Skilled and Unskilled Labor

In the baseline model, we assume that each worker supplies one unit of homogeneous labor, which can be employed either in production or in innovation. In this section, we extend the framework by introducing two types of labor: *skilled workers*, who are specialized in R&D activities, and *unskilled workers*, who are specialized in production.

Each household is endowed with labor l, drawn from a continuous distribution $\mathcal{G}(l)$ with mean normalized to unity, i.e., $\mathbb{E}[l] = 1$. A fraction Ω of this labor endowment is *skilled*, while the remaining fraction $(1 - \Omega)$ is *unskilled*. Households supply both types of labor inelastically at market wages $W^s(t)$ for skilled labor and $W^u(t)$ for unskilled labor. Similar to the baseline model, the present value budget constraint for a household with labor endowment l is modified to:

$$\int_0^\infty e^{-R(t,0)} \left[\int_0^{N(t)} P(i,t)C(i,t)di + P^X(t)X(t) \right] dt \leq \quad (1-\Omega) \int_0^\infty W^u(t)l \cdot e^{R(t,0)} dt + \Omega \int_0^\infty W^s(t)l \cdot e^{R(t,0)} dt + \mathbb{V}(0)l.$$

The supply side is the same as in the baseline model, except that only *unskilled workers* are employed in production—that is, in the production of intermediate goods and luxury services. By contrast, all three innovation activities use *skilled workers* as inputs: incumbent R&D, firm entry, and idea production. We define detrended wages as

$$w^u = e^{-(1-\gamma)g_n t} W^u(t), \quad w^s = e^{-(1-\gamma)g_n t} W^s(t).$$

Under this alternative model with two types of labor, we obtain a counterpart of Proposition 1, which characterizes the mathematical properties of the system.

Proposition C.1 *The BGE can be characterized by two functions,* $\{v(\tilde{i}), \mathcal{L}(\tilde{i})\}$, $\tilde{i} \in (0,1]$, *and five undetermined parameters,* $\{w^s, w^u, \eta, y, g_n\}$, *which jointly satisfy the following system:*

(i) The functions $v(\tilde{i})$ *solves the same ODE as in Proposition 1, subject to the boundary conditions*

$$v(1) = \frac{w^s}{\psi_e \zeta^{\theta \frac{1-\beta}{\beta}}}.$$

Here, $h(\cdot)$ is given by

$$h(g_n) = \frac{w^s \left(\theta \frac{1-\beta}{\beta}\right)^{\frac{1-\alpha}{\alpha}}}{\left[\left(\lambda^{\theta \frac{1-\beta}{\beta}} - 1\right)\psi\right]^{\frac{1}{\alpha}} \alpha} \, g_n^{\frac{1-\alpha}{\alpha}}.$$

(ii) There exists a constant $\eta > 0$ denoting the share of product varieties universally consumed across all households. If $\tilde{i} \geq \eta$, $\mathcal{L}(\tilde{i})$ satisfies

$$\begin{aligned}\mathcal{L}'(\tilde{i}) \quad = \quad & \frac{w^{u 1-\beta}}{y\beta^{\beta}(1-\beta)^{1-\beta}} \left\{ \left[\rho + \delta + \left((\sigma-1)(1-\gamma) + 1 - \theta\frac{1-\beta}{\beta}\right) g_n\right] v(\tilde{i}) \right. \\ & \left. + \, w^s \left[\frac{w^s}{\left(\lambda^{\theta\frac{1-\beta}{\beta}} - 1\right) v(\tilde{i}) \cdot \psi\alpha}\right]^{\frac{1}{\alpha-1}} \right\}^{\beta},\end{aligned}$$

subject to the same boundary conditions as in Proposition 1.

(iii) w^u is determined by the unskilled labor market clearing condition; w^s is determined by the skilled labor market clearing condition; other parameters are determined in the same way as in Proposition 1.

The proof follows almost identically to that of Proposition 1, with the main difference arising from the segmented labor market. For *unskilled workers*, the market clearing condition is

$$1 - \Omega \; = \; \int_0^1 l(\tilde{i}) \, d\tilde{i} \; + \; L^X.$$

For *skilled workers*, the market clearing condition is

$$\Omega \; = \; \int_0^1 l^I(\tilde{i}) d\tilde{i} \; + \; L^E \; + \; L^N.$$

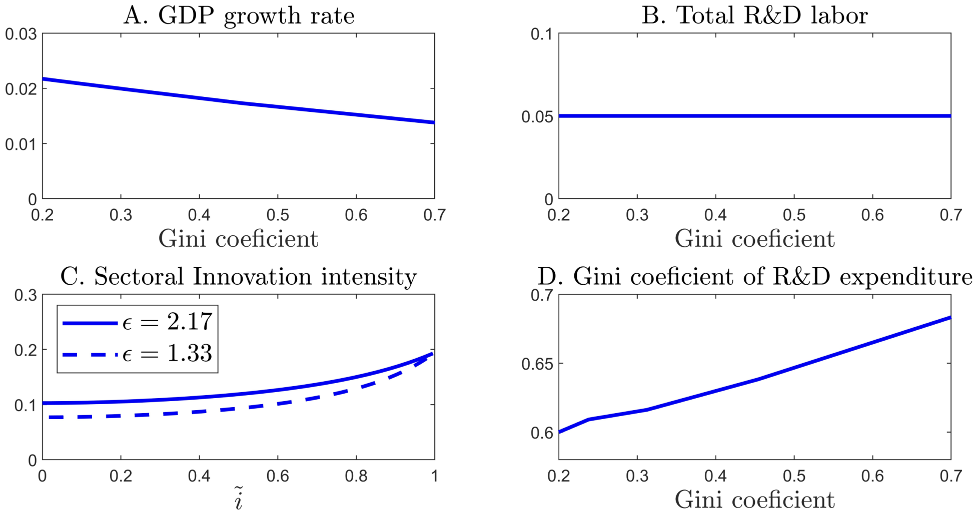
Moreover, it is straightforward to verify that the conclusions of Proposition 2 and Proposition 3 remain valid under this alternative model setup.

Following the main text, we provide a numerical example to illustrate how income inequality affects innovation. The results are shown in Figure C.1. Since the total supply of *skilled workers* is fixed at $\Omega = 0.05$, the total R&D labor is also fixed at 0.05, as displayed in Panel B. The impact of income inequality on R&D allocation and growth remains qualitatively the same as in the baseline model.

C.2 Extension: Constant Returns to Scale in Final-Good Production

In the baseline model, the production of final goods exhibits decreasing returns with respect to inputs. In this section, following Akcigit and Kerr (2018), we consider an alternative specification in which the final-good sector operates under constant returns to scale technology. Specifically, the

Figure C.1: Inequality and Growth in Alternative Model Setup



Notes. The parameters used in this example are: $\Omega = 0.05, \rho = 0.02, \sigma = 2, \gamma = 0.1, \beta = 0.1, \theta = 0.2, \delta = 0.06, \alpha = 0.5, \xi = 1.4, \iota = 0.9, \psi_e = 1, \psi = 1, \lambda = 1.2, \phi = 2, \zeta = 1$. We vary the Pareto exponent ϵ to generate different levels of inequality, targeting a Gini coefficient in the range 0.2-0.7. In Figure C, the solid line corresponds to $\epsilon = 2.17$ (Gini coefficient 0.3), while the dashed line corresponds to $\epsilon = 1.33$ (Gini coefficient 0.6).

representative final-good producer has the following production dunction:

$$Y(i,t) = \frac{\left(\bar{A}(t)L^f(i,t)\right)^{\beta}}{1-\beta} \int_0^{F(i,t)} M_i(j,t)^{1-\beta} dj.$$

Following a similar derivation in Appendix B.2.1, the profit of an intermediate-good producer is given by

$$\Pi_i(j,t) = \beta(1-\beta)^{\frac{1-\beta}{\beta}} \left[\frac{P(i,t)^{\frac{1}{1-\beta}} L^f(i,t)^{\frac{\beta}{1-\beta}}}{W(t)} \bar{A}(t)^{1-\theta} \tilde{A}_i(j,t)^{\theta} \right]^{\frac{1-\beta}{\beta}}, \tag{C.1}$$

and under constant returns to scale, the final-good producer earns zero profits in equilibrium, i.e., $\Pi(i,t) = 0$. To maintain incentives for idea creation, we assume that the idea producer receives a fraction Ψ of the profits generated by all intermediate-good producers. The value of an idea can therefore be written as

$$V^F(i,t) = \Psi \int_t^{\infty} e^{-r(\tau - t)} \beta(1-\beta)^{\frac{1-2\beta}{\beta}} P(i,\tau)^{\frac{1}{\beta}} L^f(i,\tau) W(\tau)^{-\frac{1-\beta}{\beta}} B(i,\tau) \, d\tau.$$

Apart from these modifications, the rest of the environment remains the same as in the baseline model. Under this alternative specification, we obtain a counterpart of Proposition 1 that characterizes the BGP of the economy.

Proposition C.2 *The BGP can be characterized by two functions, $\{v(\tilde{i}), \mathcal{L}(\tilde{i})\}$, $\tilde{i} \in (0,1]$, and four undetermined parameters, $\{w, \eta, y, g_n\}$, which jointly satisfy the following system:*

(i) The function $v(\tilde{i})$ solves the same ODE subject to the same boundary conditions as in Proposition 1.

(ii) There exists a constant $\eta > 0$ denoting the share of product varieties universally consumed across all households. If $\tilde{i} \geq \eta$, $\mathcal{L}(\tilde{i})$ satisfies

$$\begin{aligned} \mathcal{L}'(\tilde{i}) \quad = \quad & \frac{w^{\frac{1}{1+\beta}}}{y\beta^{\frac{2\beta}{1+\beta}}(1-\beta)^{\frac{1-\beta}{1+\beta}}\left[1-\mathcal{G}(\mathcal{L}(\tilde{i}))\right]^{\frac{\beta}{1+\beta}}} \cdot \Bigg\{ \Big[\rho+\delta+\Big((\sigma-1)(1-\gamma)+1-\theta\frac{1-\beta}{\beta}\Big)g_n\Big]v(\tilde{i}) \\ & +w \cdot \left[\frac{w}{\big(\lambda^{\theta\frac{1-\beta}{\beta}}-1\big)v(\tilde{i})\cdot\psi\alpha}\right]^{\frac{1}{\alpha-1}} \Bigg\}^{\frac{\beta}{1+\beta}}. \end{aligned} \tag{C.2}$$

subject to the same boundary conditions as in Proposition 1.

(iii) The parameters are determined in the same way as in Proposition 1.

The proof follows closely that of Proposition 1, with the main difference arising in the derivation of equation (C.2) rather than equation (23). This difference stems from the modified profit function for intermediate-good producers, that is, equation (C.1) instead of equation (B.1). To derive equation (C.2), note that the labor demand of the final-good producer satisfies

$$L^f(i,t) = \frac{\beta P(i,t)Y(i,t)}{W(t)}.$$

Substituting this expression into equation (C.1) yields a detrended HJB equation analogous to (B.35):

$$\begin{aligned} \left[\rho+((\sigma-1)(1-\gamma)+1)g_n+\delta\right]v(\tilde{i}) \quad = \quad & \beta(1-\beta)^{\frac{1-\beta}{\beta}} \cdot p(\tilde{i})^{\frac{1}{\beta}} \cdot \frac{\beta p(\tilde{i})\left[1-\mathcal{G}(\mathcal{L}(\tilde{i}))\right]}{w} \cdot w^{-\frac{1-\beta}{\beta}} \\ & -w\left(\frac{z(\tilde{i})}{\psi}\right)^{\frac{1}{\alpha}} + z(\tilde{i})\left(\lambda^{\theta\frac{1-\beta}{\beta}}-1\right)v(\tilde{i}) \\ & -\tilde{i}g_n v'(\tilde{i}). \end{aligned} \tag{C.3}$$

equation (C.2) then follows from a sequence of derivations analogous to those in Appendix B.3.3. Moreover, it is straightforward to verify that the conclusions of Proposition 2 and Proposition 3 remain valid under this alternative specification.

D Appendix: Numerical Algorithm

We solve for the balanced growth equilibrium by exploiting the structural properties characterized in Proposition 1. In particular, Proposition 1 implies that the BGE can be summarized by two functions, $\{v(\tilde{i}), \mathcal{L}(\tilde{i})\}$ for $\tilde{i} \in (0, 1]$, together with four endogenous variables, $\{w, \eta, y, g_n\}$. Numerically, this characterization turns the equilibrium problem into a shooting problem. In general, one needs to guess the initial conditions for the two differential equations at the point $\tilde{i} = 1$, together with the four aggregate variables $\{w, \eta, y, g_n\}$, and then solve the implied system so that the two ODEs, their boundary conditions, and the four equilibrium conditions in Proposition 1 are jointly satisfied. This yields a six-dimensional nonlinear system.

In practice, the problem can be reduced to a four-dimensional nonlinear system. First, the boundary value $v(1)$ is pinned down directly by the boundary condition in Proposition 1, and therefore does not need to be guessed in the shooting procedure. Second, the equilibrium system exhibits a homogeneity property with respect to y: some variables are homogeneous of degree one in y, while others are homogeneous of degree zero. In particular, if $(\{v(\tilde{i}), \mathcal{L}(\tilde{i})\}_{\tilde{i}\in(0,1]}, w, \eta, y, g_n)$ satisfies all the conditions in Proposition 1, then for any scalar $c > 0$,

$$(\{cv(\tilde{i}), \mathcal{L}(\tilde{i})\}_{\tilde{i}\in(0,1]}, cw, \eta, cy, g_n)$$

also satisfies all the conditions in Proposition 1, except for equation (B.65), which pins down the level of y. We therefore normalize $y = 1$ and solve for $\{v(\tilde{i}), \mathcal{L}(\tilde{i})\}_{\tilde{i}\in(0,1]}$ together with the remaining aggregate unknowns, leaving equation (B.65) to be imposed afterward. Once the normalized solution is obtained, we recover the equilibrium value y^* from equation (B.65), and rescale the value function and the wage according to

$$v^*(\tilde{i}) = y^* v(\tilde{i}), \qquad w^* = y^* w,$$

while leaving $\mathcal{L}(\tilde{i})$, η, and g_n unchanged.

We now describe the numerical algorithm used to solve for the equilibrium.

Step 1: Set $y = y^0 = 1$.

Step 2: Construct a four-dimensional nonlinear system in $(\bar{l}, w, \eta, g_n)$.
For any candidate vector $(\bar{l}, w, \eta, g_n)$, we define the mapping

$$\mathbf{F}(\bar{l}, w, \eta, g_n) = \begin{pmatrix} F_1(\bar{l}, w, \eta, g_n) \\ F_2(\bar{l}, w, \eta, g_n) \\ F_3(\bar{l}, w, \eta, g_n) \\ F_4(\bar{l}, w, \eta, g_n) \end{pmatrix},$$

where the four residuals are given by:

(i) *Boundary condition of $\mathcal{L}(\tilde{i})$.*

For any candidate vector $(\bar{l}, w, \eta, g_n)$, we first solve the ODE system characterized by equations (22) and (23) in Proposition 1, subject to the boundary conditions

$$v(1) = \frac{w}{\psi_e \zeta^{\theta \frac{1-\beta}{\beta}}}, \qquad \mathcal{L}(1) = \bar{l}.$$

This yields the function $v(\tilde{i})$ and $\mathcal{L}(\tilde{i})$. We then evaluate whether the implied value of $\mathcal{L}(\tilde{i})$ at $\tilde{i} = \eta$ satisfies the boundary condition in Proposition 1. Accordingly, the first residual is defined as

$$F_1(\bar{l}, w, \eta, g_n) = \mathcal{L}(\eta) - \underline{l}.$$

(ii) *Labor market clearing condition.*

Using $v(\tilde{i})$ and $\mathcal{L}(\tilde{i})$, we recover the remaining sectoral objects pointwise. In particular, we obtain $p(\tilde{i})$ from equation (B.48), $b(\tilde{i})$ from equation (B.38), $v^f(\tilde{i})$ from equation (B.42), $f(\tilde{i})$ from equation (B.36), and $e(\tilde{i})$ from equation (B.37). These objects in turn determine labor demand in production, incumbent R&D, entry, luxury services, and new-good invention, denoted by $l(\tilde{i})$, $l^I(\tilde{i})$, L^E, L^X, and L^N, respectively, as characterized by equations (B.51)–(B.55). The second residual is then defined by the labor market clearing condition:

$$F_2(\bar{l}, w, \eta, g_n) = \int_0^1 \left[l(\tilde{i}) + l^I(\tilde{i}) \right] d\tilde{i} + L^E + L^X + L^N - 1.$$

(iii) *Aggregate resource constraint.*

Using the detrended aggregate resource constraint in equation (B.58), we define the third residual as

$$F_3(\bar{l}, w, \eta, g_n) = y(1-\beta)^2 \left[1 - \int_{\mathcal{L}(1)}^{\infty} \big(l - \mathcal{L}(1) \big) \, d\mathcal{G}(l) \right] - w \int_0^1 l(\tilde{i}) \, d\tilde{i}.$$

(iv) *Entry condition for final-good producers.*

Using the detrended entry condition for final-good producers in equation (B.59), we define the fourth residual as

$$F_4(\bar{l}, w, \eta, g_n) = w - \phi v^f(1).$$

Step 3: Solve the nonlinear system

$$\mathbf{F}(\bar{l}, w, \eta, g_n) = 0$$

using the robust Newton method. Denote the solution by

$$(\bar{l}^0, w^0, \eta^0, g_n^0),$$

and the corresponding functions by $\{v^0(\tilde{i}), \mathcal{L}^0(\tilde{i})\}_{\tilde{i}\in(0,1]}$. Together with the normalization $y^0 = 1$, these functions and variables satisfy all the conditions in Proposition 1, except for equation (B.65), which pins down the level of y through the price normalization condition.

Step 4: Use equation (B.65) to recover the equilibrium level of y and rescale the normalized solution obtained in **Step 3**.

Using equation (B.65), we update y from its normalized value $y^0 = 1$ to

$$y^1 = \int_{\underline{l}}^{\mathcal{L}^0(1)} \frac{\left[(\mathcal{L}^{0^{-1}}(l))\right]^{1-\gamma}}{1-\gamma}\, d\mathcal{G}(l) + \int_{\mathcal{L}^0(1)}^{\infty} \frac{1}{1-\gamma}\, d\mathcal{G}(l) + \frac{g_n^0}{\phi}.$$

The final equilibrium solution is then given by

$$\begin{cases} v^1(\tilde{i}) = y^1 v^0(\tilde{i}), & \tilde{i} \in (0,1], \\ \mathcal{L}^1(\tilde{i}) = \mathcal{L}^0(\tilde{i}), & \tilde{i} \in (0,1], \\ w^1 = y^1 w^0, \\ \eta^1 = \eta^0, \\ g_n^1 = g_n^0, \\ y^1 \text{ as given above.} \end{cases}$$

E Appendix: Quantitative Exercise

This appendix provides additional materials for the quantitative exercise. We present several technical derivations and discuss further robustness checks to support the quantitative results reported in the main text.

E.1 Dynamics of Firm Distribution

We characterize the evolution of the within-sector firm distribution by tracking firms' positions on a discrete productivity ladder. When a product is newly created, all firms start from the same productivity level $\zeta\bar{A}(t)$, where $\bar{A}(t)$ denotes the aggregate productivity level. Thereafter, productivity improvements arise only through innovation. Each successful innovation raises firm productivity by a factor $\lambda > 1$. Therefore, a firm's productivity is fully determined by the number of successful innovations it has accumulated.

Let $j \in \{0, 1, \dots\}$ denote the number of successful innovations, which is truncated for numerical purpose. Then firm productivity at ladder step j is given by

$$\tilde{A}_i(j,t) = \zeta\bar{A}(t)\lambda^j.$$

Productivity is a state variable, and the cross-sectional distribution of firms over j fully characterizes the distribution of firm productivity. In turn, this distribution is sufficient to determine equilibrium objects such as firm size, and firm level R&D expenditure.

Let $M_i(j,t)$ denote the mass of firms that have experienced j successful innovations at time t in sector i. The law of motion for the distribution is given by the following system of differential equations. For the lowest productivity level $j = 0$,

$$\dot{M}_i(0,t) = \left(\frac{e(i,t)}{F(i,t)} - \delta\right) M_i(0,t) - z_i(t)M_i(j,t),$$

which reflects firm entry, firm exit, and innovation to higher ladders. For higher productivity levels $j = 1, 2, \dots,$

$$\dot{M}_i(j,t) = \left(\frac{e(i,t)}{F(i,t)} - \delta\right) M_i(j,t) + z_i(t)\left[(M_i(j-1,t) - M_i(j,t)\right].$$

The first term captures firm entry and exit, while the second term describes innovation-driven redistribution. Firms at level $j-1$ move to level j upon successful innovation, while firms at level j move further up the ladder. This system defines a linear transport equation over the productivity ladder, where innovation induces an upward flow of firms across grid points.

Numerical implementation. An important simplification is that the computation of the balanced growth equilibrium does not require firm-level information. In particular, the equilibrium objects that characterize the BGE—including the innovation intensity $z_i(t)$, firm entry $e(i,t)$, and the total mass of firms $F(i,t)$—are determined entirely at the sectoral level. We therefore solve for the full equilibrium first, and then use the implied paths of $z_i(t)$, $e(i,t)$, and $F(i,t)$ as inputs to characterize the within-sector firm distribution.

Let $j \in \{0, 1, \dots, N_{\text{grid}} - 1\}$ denote the number of successful innovations, where the state space is truncated at N_{grid} for numerical implementation. For a sector i^* invented at time t^*, the state vector

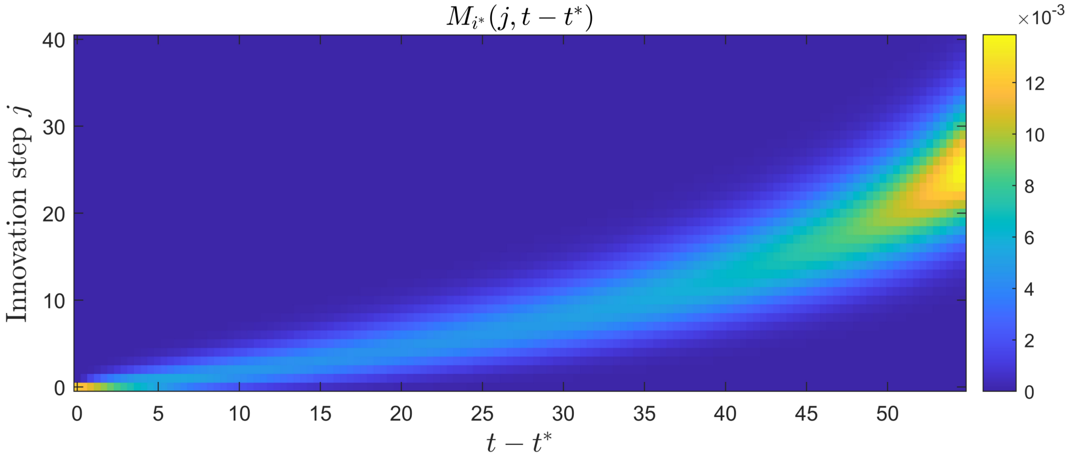
$$M_{i^*}(t) = \left[M_{i^*}(0,t), \dots, M_{i^*}(N_{\text{grid}} - 1, t)\right]'$$

evolves according to the ODE system above. Conditional on the equilibrium paths of $z_{i^*}(t)$, $e(i^*,t)$, and $F(i^*,t)$, together with the boundary condition

$$M_{i^*}(t^*) = \left[F(i^*, t^*), 0, \dots, 0\right]',$$

we solve this system using standard ODE solvers and obtain the implied dynamics of the firm distribution along the balanced growth path. As shown in Figure E.1, with firm entry and innovation accumulate over time, the distribution gradually shifts toward higher productivity ladders.

Figure E.1: Firm Distribution Dynamics



Notes. The figure illustrates the evolution of $M_{i^*}(t)$ under the calibrated parameter values.

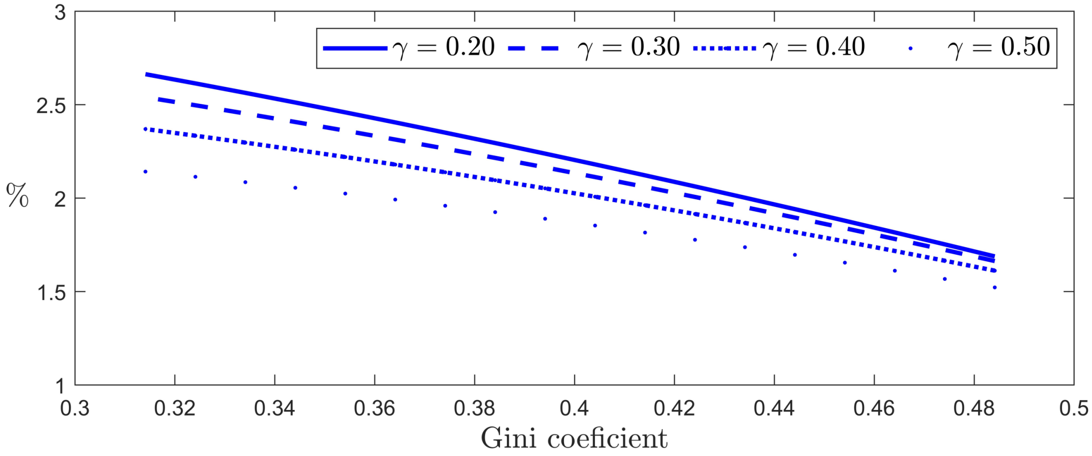
E.2 Quantitative Result with Different γ

This appendix examines the sensitivity of our quantitative results to the preference curvature parameter γ, which governs the degree of non-homotheticity in household demand. In the baseline calibration, we set $\gamma = 0.343$ to match the observed average price growth rate of "necessities" in our sample period. To assess robustness, we recompute the main counterfactual exercise under alternative values of $\gamma \in \{0.20, 0.30, 0.40, 0.50\}$, holding all other parameters fixed.

Figure E.2 reports the implied per-capita growth rates with different inequality levels under each value of γ. Two patterns emerge. First, varying γ mainly affects the level of growth, reflecting differences in the strength of income-driven demand reallocation across goods. Second and more importantly, the relative decline in growth resulting from rising inequality is remarkably stable across specifications. For all values of γ, the model continues to generate a sizable slowdown in aggregate growth when the income distribution shifts toward greater inequality.

Quantitatively, the relative contribution of rising inequality to the observed growth slowdown remains similar across specifications, indicating that our main quantitative conclusions are not driven by a particular choice of γ. Overall, these results show that the mechanism linking inequality, innovation incentives, and long-run growth is robust to plausible variation in the degree of demand non-homotheticity.

Figure E.2: Per-capita Growth with Different value of γ



Notes. This figure compares model-implied growth rates with different inequality levels under different value of γ.

E.3 Distribution of After-tax Income

Let pre-tax labor income be $l \geq \underline{l} > 0$ with Pareto density

$$f(l) = \frac{\epsilon\, \underline{l}^{\,\epsilon}}{l^{\epsilon+1}}, \qquad \epsilon > 0,$$

and consider the balanced-budget tax schedule

$$T(l) = l - \frac{l^{1-\upsilon}}{\displaystyle\int_{\underline{l}}^{\infty} l^{1-\upsilon}\, d\mathcal{G}(l)} = l - \frac{l^{1-\upsilon}}{\displaystyle\int_{\underline{l}}^{\infty} l^{1-\upsilon}\, \frac{\epsilon\, \underline{l}^{\,\epsilon}}{l^{\epsilon+1}}\, dl}, \qquad \upsilon \in [0,1], \tag{E.1}$$

where $\mathcal{G}$ is the CDF of l. The denominator equals

$$\int_{\underline{l}}^{\infty} l^{1-\upsilon}\, d\mathcal{G}(l) = \epsilon\, \underline{l}^{\,\epsilon} \int_{\underline{l}}^{\infty} l^{-(\epsilon+\upsilon)}\, dl = -\epsilon\, \underline{l}^{\,\epsilon} \left[\frac{l^{\,1-(\epsilon+\upsilon)}}{\epsilon+\upsilon-1} \right]_{l=\underline{l}}^{\infty} = \frac{\epsilon}{\epsilon+\upsilon-1}\, \underline{l}^{\,1-\upsilon}.$$

Hence after-tax income $Y \equiv l - T(l)$ is

$$Y = \frac{l^{1-\upsilon}}{\displaystyle\int l^{1-\upsilon} d\mathcal{G}(l)} = \underbrace{\frac{\epsilon + \upsilon - 1}{\epsilon}}_{C} \left(\frac{l}{\underline{l}} \right)^{1-\upsilon}, \tag{E.2}$$

with constant $C \equiv (\epsilon + \upsilon - 1)/\epsilon$.

For $\upsilon \in [0, 1)$, the mapping $l \mapsto Y$ in (E.2) is strictly increasing. The support of Y is $[C, \infty)$ since $Y = C$ at $l = \underline{l}$. Using a monotone transformation,

$$\Pr(Y \leq y) = \Pr\left(l \leq \underline{l} \left(\tfrac{y}{C} \right)^{\frac{1}{1-\upsilon}} \right) = 1 - \left(\frac{\underline{l}}{\underline{l} \left(y/C \right)^{1/(1-\upsilon)}} \right)^{\epsilon} = 1 - \left(\frac{C}{y} \right)^{\frac{\epsilon}{1-\upsilon}}.$$

Therefore, the CDF and PDF of Y are

$$F_Y(y) = 1 - \left(\frac{C}{y} \right)^{\frac{\epsilon}{1-\upsilon}}, \qquad f_Y(y) = \frac{\epsilon}{1-\upsilon} \frac{C^{\frac{\epsilon}{1-\upsilon}}}{y^{\frac{\epsilon}{1-\upsilon}+1}}. \tag{E.3}$$

i.e., $Y \sim \text{Pareto}\left(C, \frac{\epsilon}{1-\upsilon} \right)$. When $\upsilon = 0$ (no redistribution), $Y = C(l/\underline{l}) = l$ and the shape parameter is ϵ (the pre-tax shape), as expected. When $\upsilon \to 1$, $Y \to 1$ for all l, and the distribution degenerates at a constant (perfect equality). Moreover, the mean of Y exists and equals one:

$$\mathbb{E}[Y] = C \cdot \frac{\frac{\epsilon}{1-\upsilon}}{\frac{\epsilon}{1-\upsilon} - 1} = \frac{\epsilon + \upsilon - 1}{\epsilon} \cdot \frac{\epsilon}{\epsilon + \upsilon - 1} = 1,$$

confirming that the schedule (E.1) is balanced-budget for any $\upsilon \in [0, 1)$ with $\epsilon + \upsilon > 1$.